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Proportional Values for Cooperative Games

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Introduction

Cooperative game theory

Game theory uses mathematical models to explore situations with conflict and cooperation between decision makers (players). The seminal book "*Theory of Games and Economic Behavior*" by von Neumann and Morgenstern (1944) is considered as a milestone of the modern theory of games. In this book, two fundamentally different approaches are distinguished to analyze potentially complex patterns of strategic behavior. The first approach is known as non-cooperative game theory and targets the interactions between players that result from their chosen strategies. The second approach is known as cooperative game theory and targets interactions caused by cooperation in which players can make fully binding and enforceable agreements.

Instead of modelling explicitly the actions that players must take to carry out these agreements, a cooperative game is an austere model derived from strategic situations in which the opportunities available to each coalition of players could be described by a single number. These numbers can be considered as a joint utility or payoff for the players in the coalitions. The assumption that the utility transfers are possible among the players leads to the class of cooperative games with transferable utility, abbreviated as TU-games. Formally, a TU-game consists of a set of *players* and a *characteristic function* that specifies a worth to each coalition of players. As stated by Roth (1988), although these simplifying assumptions are obviously substantial, the TU-game model has proved to be surprisingly useful as a simple model of strategic interaction. In cooperative game theory, an essential issue is to build a universal value (or solution) concept that describes the distribution of the attainable gains among the cooperating players in a reasonable way for every game. The problem arises of "*How? What formula or procedure should we use? On what principles should we base their evaluation?*" (Hokari and Thomson, 2015). Unlike the Nash equilibrium (Nash, 1950) in non-cooperative game theory, *no* value concept has emerged as a leading notion in cooperative game theory that would satisfy everyone's preferences. It, in some sense, makes this field more fascinating.

Values for TU-games are usually supported by axiomatizations or axiomatic characterizations. Axiomatization consists of two steps: first, formulate desirable properties of values, as axioms; second, identify the values satisfying the properties in various combinations. This normative method for TU-games has been pioneered by Shapley (1953a), who axiomatically introduced the Shapley value by efficiency, additivity, symmetry, and the null player property.

Fairness: equality and proportionality

Justice is blind, and fairness requires anonymous rules of arbitration (Moulin, 2004). Typically, fairness does not receive a unique interpretation in allocation problems, we refer to Young (1995), Brams and Taylor (1996), Moulin (2004), Feldman and Serrano (2006), Fleurbaey (2008), Hougaard (2009), Fleurbaey and Maniquet (2011), Thomson (2011b), Kaufman (2018), and Thomson (2019). Consequently, the theory of fair allocation aims to investigate possible ways of capturing intuitions of fairness, formulate axioms that encapsulate these intuitions, and identify allocation rules that satisfy the axioms.

As stated by Young (1995), the oldest and most prominent in discussions about distributive fairness is Aristotle's celebrated maxim, which says "Equals should be treated equally, unequals unequally, in proportion to relevant similarities and differences" (Nicomachean Ethics). Two alloction principles are deciphered: *equal treatment of equals* and *unequal treatment of unequals*. *Equal treatment of equals* is an equality principle, which states that if two players have identical characteristics in the allocation problem at hand, they should receive the same treatment. *Unequal treatment of unequals*, by contrast, is a proportionality principle, which asserts that the total worth sould be divided in proportion to each player's contribution. The former is a clear-cut principle, whereas the latter is a vague principle since in some situations the measure of contribution is not clear.

Equality and proportionality are often taken as notions of fairness in the value theory for TU-games. This even can be reflected by the Shapley value (Shapley, 1953a) and the weighted Shapley value (Shapley, 1953b). The Shapley value allocates the Harsanyi dividends (Harsanyi, 1959) equally among all players in a coalition. The idea of equality is also captured by symmetry (also known as equal treatment of equals in Hart (1990)), one of its standard axioms, which states that identical players have equal payoffs. In contrast, the weighted Shapley value allocates the Harsanyi dividends proportionally to the exogenous weights of players among all players in a coalition. Characterizing the weighted Shapley value, various proportionality axioms were introduced by modifying the symmetry axiom, we refer to Kalai and Samet (1987), Chun (1991), Nowak and Radzik (1995), Casajus (2018), and Casajus (2019).

In fact, equality and proportionality are notions that play a prominent role in almost all branches of the theory of economic justice. In some resource allocation problems that are closely related to TU-games, in particular bankruptcy problems and claims problems, proportionality seems more appropriate and is widely accepted. The seminal work was done by O'Neill (1982), who takes an axiomatic approach to the proportional rule of the bankruptcy problems in which agents have claims on the estate of a single bankrupt agent. Subsequently, notable axiomatizations of the proportional rule for bankruptcy problems are given by, e.g. Young (1988), Chun (1988), Ju et al. (2007a), Moreno-Ternero (2006), Giménez-Gómez and Peris (2014), and Flores-Szwagrzak et al. (2020). Some related works also have been done. For example, Young (1987) studies a class of (symmetric and asymmetric) parametric rules for claims problems, and this class includes the equal and proportional rules; Moulin (1991) discusses the equal and proportional solutions for the surplus-sharing problems; Moulin (2016) considers proportional assignment and rationing of goods with different characteristic; Izquierdo and Timoner (2019) study the proportional rule for decentralized rationing problems in which the resource is not directly assigned to agents, but first allocated to groups of agents and then divided among their members; Ghamami and Glasserman (2019) analyze the optimal allocation of trades to portfolios when the cost associated with an allocation is proportional to each portfolio's risk; Eisenberg and Noe (2001) extend the proportional rule for claims problems to the set-up of financial networks; Csóka and Herings (2021) firstly provide an axiomatization of the proportional rule in financial networks; Leshno and Strack (2020) characterize the proportional rule for a decentralized network of anonymous computers.

Despite their auspicious beginnings and flourishing in allocation problems, proportionality has been used far less than equality as an established principle in TU-games. This imbalance may seem strange. *Why*? Perhaps one answer is that proportionality is much less obvious in TU-games. Moreover, proportionality is translated into mathematical requirements sensibly depending on the specifications of the problem at work.

Recently, there has been a growing body of the literature on proportionality in TU-games. Various proportional values have been considered. Some are defined naturally associating with the exogenous weights of players, we refer to the weighted division value (Béal et al., 2016b), the weighted surplus division value (Calleja and Llerena, 2017; Calleja and Llerena, 2019), and the weighted ENSC value (Hou et al., 2019). Others are introduced based on characteristics of TU-games, we refer to the proportional value (Ortmann, 2000; Khmelnitskaya and Driessen, 2003; Kamijo and Kongo, 2015), the proper Shapley values (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015; van den Brink et al., 2020), the proportional Shapley value (Béal et al., 2018; Besner, 2019), and the proportional Harsanyi solution (Besner, 2020).

This thesis mainly studies values based on the proportionality principles in TU-games, and also compares and combines them with the equality principles. An overview of each chapter of the thesis is explained in the next section.

Overview of the thesis

The main contribution of this thesis is to provide axiomatizations of proportional values for TU-games. The thesis contains six chapters. Chapter 1 gives an introduction to the main concepts, definitions, and notation about TU-games. The chapters

2–5 aim at providing new values for some classes of TU-games, and then characterizing those proposed values. Chapter 6 is devoted to axiomatizing a known and related family of egalitarian values for TU-games.

In Chapter 2, we present axiomatic characterizations of the *proportional division (PD) value* for TU-games, which allocates the worth of the grand coalition in proportion to the stand-alone worths of players. First, a new proportionality principle, called *proportional-balanced treatment*, is introduced by strengthening Shapley's symmetry axiom, which states that if two players make the same contribution to any nonempty coalition, then they receive amounts in proportion to their stand-alone worths. We characterize the family of values satisfying efficiency, weak linearity, and proportional-balanced treatment. We also show that this family is incompatible with the dummy player property. However, we show that the PD value is the unique value in this family that satisfies the dummifying player property. Second, we propose three appropriate monotonicity axioms by considering two games in which the stand-alone worths of all players are equal or in the same proportion to each other, and obtain three axiomatizations of the PD value without both weak linearity and the dummifying player property. Third, from the perspective of a variable player set, we show that the PD value is the only one that satisfies proportional standardness and projection consistency. Finally, we provide characterizations of proportional standardness.

Chapter 3 studies the implications of extending the balanced cost reduction property from queueing problems[1] to general TU-games. For queueing problems, *balanced cost reduction* together with *efficiency* and *Pareto indifference* characterize the *minimal transfer rule*, being one of the most popular values for queueing problems, which is obtained by applying the Shapley value to an associated TU-game. Since queueing games are so-called *2-games*, the minimal transfer rule coincides with other TU-game values, such as the pre-nucleolus and the τ-value of the associated queueing game. As a direct translation of the balanced cost reduction property, the axiom of *balanced externalities* for values of TU-games, requires that the payoff of any player is equal to the total externality it inflicts on the other players with its presence. We show that this axiom and efficiency together characterize the Shapley value for 2-games. However, extending this axiom in a straightfoward way to general TU-games is incompatible with efficiency. Keeping as close as possible to the idea behind balanced externalities, we weaken this axiom by requiring that every player's payoff is the same *fraction* of its total externality inflicted on the other players. This weakening, which we call *weak balanced externalities*, turns out to be compatible with efficiency. More specifically, the unique efficient solution that satisfies this weaker property is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates

[1]Consider a group of agents who must be served in a facility. The facility can handle only one agent at a time and agents incur waiting costs. The queueing problem is concerned with finding the order in which to serve agents and the (positive or negative) monetary compensations they should receive. It is known that queueing games are so-called *2-additive games*, or shortly *2-games*, meaning that the worth is fully generated by coalitions of size two. We refer to Maniquet (2003) and Chun (2006).

the total worth proportional to the *separable costs* of the players. This value is the dual of the PD value discussed in Chapter 2. Based on this duality, we provide axiomatizations of the PANSC value using a reduced game consistency axiom. Besides, we consider a comparison with the EANSC value (Moulin, 1985), as well as the *Separable Costs Remaining Benefits* (SCRB) method (Young et al., 1982) and *Alternative Cost Avoided* (ACA) method (Straffin and Heaney, 1981; Otten, 1993) in cost allocation problems.

Chapter 4 explores a new family of values for TU-games that offer a compromise between the PD value and the equal division (ED) value. Recall that the family of convex combinations of the PD and ED values (Moulin, 1987) considers only the worths of all singleton coalitions and the grand coalition. Dutta and Ray (1989) argue that all coalitions should be considered when formulating an (egalitarian) allocation in a TU-game. This is clearly not the case when one considers the ED or PD value, or any convex combination of them. As an extension of this family, our family is defined by considering not only the proportional and equal division methods, but also the worths of all coalitions. Our value, called an α-mollified value, is obtained in two steps. First, a linear function with respect to the worths of all coalitions is defined which associates a real number to every TU-game. Second, the weight assigned by this function is used to weigh proportionality and equality principles in allocating the worth of the grand coalition. We provide an axiomatic characterization of this family, and show that this family contains the affine combinations of the ED value and the equal surplus division (ESD) value as the *only* linear values. Further, we identify the PD value and the affine combinations of the ED and ESD values as those members of this family, that satisfy projection consistency. Besides, we provide a procedural implementation of each single value in our family.

Chapter 5 concentrates on a family of 'proportional sharing of the surplus' type of values for TU-games, which is a subfamily of values introduced in Chapter 4. These values are called *proportional surplus division values* and first make a trade-off between a player's stand-alone worth and the average stand-alone worth, and then allocate the remainder proportional to the stand-alone worths. This family contains the PD value and the new egalitarian proportional surplus division value as two special cases. The first one applies an egocentric principle and first assigns to each player its own stand-alone worth, whereas the second focuses on egalitarianism in allocating the stand-alone worths by first assigning to every player the average of all stand-alone worths. Both values apply proportionality in the allocation of the remaining surplus. We provide characterizations for this family of values, as well as for each single value in this family. Our characterizations involve two new axioms which evaluate the consequences of separatorization of a player in TU-games. *Separatorization* requires that a player becomes a separator, i.e. a dummifying player (Casajus and Huettner, 2014a), in a TU-game. Specifically, given a TU-game, separatorization of a player means that the worth of any coalition containing this player becomes equal to the sum of the stand-alone worths of the players in this coalition.

The first new axiom is *proportional loss under separatorization*, which states that, if a player becomes a separator, then any two other players are affected proportionally to their stand-alone worths. The second new axiom is *proportional balanced contributions under separatorization*, which states that any two players are affected proportionally to their stand-alone worths if the other becomes a separator. Notice that separatorization is in line with 'veto-ification' introduced in van den Brink and Funaki (2009), dummification introduced in Béal et al. (2018), and nullification studied in Gómez-Rúa and Vidal-Puga (2010), Béal et al. (2016b), Ferrières (2017), Kongo (2018), Kongo (2019), and Kongo (2020).

In Chapter 6, we explore the ED value, the ESD value, and the classes of affine and convex combinations of them involving the separatorization discussed in Chapter 5. We suggest a new axiom called *equal loss under separatorization*, which requires that if a player becomes a separator, then any two other players are equally affected. This axiom together with efficiency, fairness, and homogeneity characterize the class of affine combinations of the ED and ESD values. Replacing fairness with linearity and symmetry yields another axiomatization. We also show that efficiency, equal loss under separatorization, additivity, desirability, and superadditive monotonicity characterize the class of convex combinations of the ED and ESD values. Equal loss under separatorization and proportional loss under separatorization (studied in Chapter 5) respectively suppose equality and proportionality principles on allocation rules when a player becomes a separator. As a contrast to the axiomatic results in Subsection 5.4.1, we also provide alternative characterizations of the classes of affine and convex combinations of the ED and ESD values.

Chapter 1

Preliminaries

Game theory is the study of mathematical models of strategic interaction among rational players. A player is said to be rational if the player always makes decisions in pursuit of her own objectives. Based on whether the players are able to form fully binding commitments, game theory classifies into two branches: non-cooperative game theory and cooperative game theory. As it is well known, cooperative games capture situations in which players can make fully binding commitments to form coalitions. When it is assumed that all players choose to cooperate, the question is what should be assigned to each player? The objective of cooperative game theory is to answer this question.

In this thesis, we will focus on cooperative game theory. A situation in which a finite set of players can generate certain worths by cooperation can be discribed by a cooperative game with transferable utility, or simply a TU-game. This chapter introduces terminology of TU-games that will be used throughout the thesis. In Section 1.1, we formally introduce TU-games. In Section 1.2, we introduce several values for TU-games. In Section 1.3, we review some properties and results for these values. For the ease of the reader, we will repeat the relevant definitions in each chapter.

1.1 Cooperative games with transferable utility

Let $\mathcal{N}$ be the universe of potential players, and let $N \subset \mathcal{N}$ be a finite set of n players. The notation $S \subseteq T$ means that S is a subset of T, while the notation $S \subset T$ means that S is a proper subset of T. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of all real numbers and positive real numbers, respectively.

A *cooperative game with transferable utility*, or simply a *(TU-)game*, is a pair (N, v), where $N \subset \mathcal{N}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning a *worth* $v(S)$ to each $S \subseteq N$ such that $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition*, and $v(S)$ is the reward that coalition S can guarantee by itself without the cooperation of the other players. The cardinality of a set S will be denoted by $|S|$ or, if no ambiguity is possible, appropriate small letter s. Denote $\mathcal{G}$ as the class of all TU-games with a finite player set in $\mathcal{N}$, and $\mathcal{G}^N$ the class of TU-games with player set N. For brevity, we refer to a TU-game just as a game.

Various subclasses of games have been considered in the literature. Some have arisen naturally in applications to economics; others have emerged out of mathematical considerations. We collect some definitions of games as follows.

A game (N, v) is

- *additive* or *inessential*: if $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$.
- *superadditive*: if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.
- *monotone*: if $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$.
- *quasi-additive*: if $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \subset N$.
- *weakly essential*: if $\sum_{i\in N} v(\{i\}) \leq v(N)$.
- *null game*: if $v(S) = 0$ for all $S \subseteq N$.

The quasi-additive games (Carreras and Owen, 2013) are closely related to joint venture situations in Moulin (1987). Obviously, a quasi-additive game (N, v) reduces to an additive game if $v(N) = \sum_{i\in N} v(\{i\})$.

Given (N, v), its dual game (N, v^*) is defined by

$$v^*(S) = v(N) - v(N\backslash S) \quad \text{for all } S \subseteq N.$$

Duality can be applied to solutions as well, and to properties of solutions. For studies of duality, we refer to Charnes et al. (1978) and Oishi et al. (2016).

The *unanimity game* for a nonempty coalition $S \subseteq N$ is the game $(N, u_S) \in \mathcal{G}^N$, where u_S is defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that the collection of unanimity games $\{(N, u_S) \mid 0 \neq S \subseteq N\}$ forms a basis for $\mathcal{G}^N$, i.e., every game $(N, v) \in \mathcal{G}^N$ can be expressed by $v = \sum_{S\subseteq N, S\neq\emptyset} \Delta_v(S) u_S$, where $\Delta_v(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|} v(T)$ is the *Harsanyi dividend* (Harsanyi, 1959) of coalition S in the game.

Finally, we introduce a restrictive class of TU-games studied in Béal et al. (2018). This class plays a major role in this thesis. Following Béal et al. (2018), a game (N, v) is *individually positive* if $v(\{i\}) > 0$ for all $i \in N$, and *individually negative* if $v(\{i\}) < 0$ for all $i \in N$. Let $\mathcal{G}_{nz}$ denote the class consisting of all individually positive and individually negative games, and let $\mathcal{G}^N_{nz}$ denote the intersection of $\mathcal{G}_{nz}$ and $\mathcal{G}^N$.

The notation $\mathcal{A}^N$ (respectively $\mathcal{QA}^N$) denotes the class of all additive games (respectively quasi-additive games) in $\mathcal{G}^N$. We express the notation $\mathcal{A}^N_{nz}$ (respectively $\mathcal{QA}^N_{nz}$) for the class of all additive games (respectively quasi-additive games) in $\mathcal{G}^N_{nz}$.

We remark that Béal et al. (2018) provide many applications of this restrictive class of TU-games, such as land production economies, telecommunication problems, and sequencing/queueing problems. Thus, most of our results can also be applied to these economics problems and other related problems.

1.2 Value concepts for TU-games

Cooperative game theory focuses on two main questions: (i) what coalition will form? (ii) how to allocate the attainable worth to each player? In this thesis, we leave out the strategic aspect of coalition formation and consider the second question under the assumption that the players who are participants in a TU-game can work together to form every coalition. As mentioned in the Introduction, various value concepts for TU-games have been proposed concerning different fairness criteria. In this section, we recall several value concepts that will be further discussed in the following chapters. These values are divided into two classes, which rely on equality (or egalitarian) and proportionality principles.

A (single-valued) *solution* or a *value* on a class of TU-games $\mathcal{C} \subseteq \mathcal{G}$ is a function ψ that assigns a single payoff vector $\psi(N,v) \in \mathbb{R}^N$ to every game $(N,v) \in \mathcal{C}$. One of the most well-known values in the literature is the Shapley value introduced in Shapley (1953a). The *Shapley value* assigns to every player its expected marginal contribution assuming that all possible permutations in which the grand coalition can be formed occur with equal probability.

Definition 1.1. The *Shapley value* on $\mathcal{G}^N$ is defined by

$$Sh_i(N,v) = \sum_{S\subseteq N, i\in S} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S\backslash\{i\})]$$

for all $(N,v) \in \mathcal{G}^N$ and $i \in N$.

The Shapley value can also be represented by allocating the Harsanyi dividends equally over the players in the corresponding unanimity coalition. Namely,

$$Sh_i(N,v) = \sum_{S\subseteq N, i\in S} \frac{\Delta(S)}{|S|}.$$

In this sense, the Shapley value can be seen as satisfying a notion of equality.

Instead of considering all the coalitions, as the Shapley value does, the literature also studies several egalitarian values that only take into account some particular class of coalitions. Various examples of such values are the equal division value, the equal surplus division value, and the equal allocation of nonseparable costs value. Formally, their definitions are as follows.

The *equal division (ED) value*, axiomatized in van den Brink (2007), allocates the worth of the grand coalition equally among all players.[1]

Definition 1.2. The *ED value* on $\mathcal{G}^N$ is defined by

$$ED_i(N,v) = \frac{v(N)}{n}$$

for all $(N,v) \in \mathcal{G}^N$ and $i \in N$.

The *equal surplus division (ESD) value*, also known as *Centre-of-the-Imputation-Set (CIS) value* in Driessen and Funaki (1991), assigns to each player its own stand-alone worth and an equal share of the remainder.

Definition 1.3. The *ESD value* on $\mathcal{G}^N$ is defined by

$$ESD_i(N,v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]$$

for all $(N,v) \in \mathcal{G}^N$ and $i \in N$.

The *equal allocation of nonseparable cost (EANSC) value*[2], introduced by Moulin (1985), assigns to every player its separable cost and an equal share of the remainder. The separable cost is the contribution of a player to the grand coalition (if the utility of any coalition is a worth/payoff but not a cost, we often use nonseparable contribution instead of nonseparable cost). Formally, for all $(N,v) \in \mathcal{G}^N$ and $i \in N$, the *separable cost* is defined by

$$SC_i(N,v) = v(N) - v(N\backslash\{i\}). \tag{1.1}$$

Definition 1.4. The *EANSC value* on $\mathcal{G}^N$ is defined by

$$EANSC_i(N,v) = SC_i(N,v) + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j(N,v)] \tag{1.2}$$

for all $(N,v) \in \mathcal{G}^N$ and $i \in N$.

Notice that the EANSC value is the ESD value of the dual game.

In this thesis, the above values are often considered on other classes of games instead of the class of all TU-games. To avoid confusion, we will recall these values in each chapter.

We now turn to notions of proportionality. When allocating payoffs, a baseline should be specified for each player from which to measure her gain, but also an amount to which she may aspire. Particularly, the proportional Shapley value and the proportional division value rely on two different proportionality principles.

[1] Radzik (2013) calls it the equal split value.
[2] This value is often denoted by EANS value or ENSC value.

The *proportional Shapley value*, studied in Béal et al. (2018) and Besner (2019), allocates the Harsanyi dividends proportionally to the stand-alone worths among all players in the corresponding coalition.

Definition 1.5. The *proportional Shapley value* on $\mathcal{G}^N_{nz}$ is defined by

$$PSh_i(N,v) = \sum_{S \subseteq N, i \in S} \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} \Delta(S)$$

for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

The *proportional division (PD) value* allocates the worth of the grand coalition proportionally to the stand-alone worths among all players.

Definition 1.6. The *PD value* on $\mathcal{G}^N_{nz}$ is defined by

$$PD_i(N,v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$$

for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

In this thesis, the PD value is often considered on other classes of games instead of $\mathcal{G}^N_{nz}$. To avoid confusion, we will recall it in each chapter.

The PD value is identical to the proportional rule in Moriarity (1975) and Banker (1981), and the stand-alone-coalition proportional value in Kamijo and Kongo (2015). We remark that the PD value cannot be considered as a weighted division value (Béal et al., 2016b) or the weighted surplus division value (Calleja and Llerena, 2017; Calleja and Llerena, 2019) since those values are based on exogenous weights, while the weights in the PD value are determined in the game, specifically they are equal to the stand-alone worths.

Notice that the PD value often appears in the literature as an example to show the logical independence of axioms of values for TU-games. However, an axiomatic characterization of the PD value for general TU-games is still missing.[3] This also motivates us to study the PD value and other related values.

1.3 Properties of values

There is no consensus which is the unique 'best' value in cooperative game theory. One should not expect to find a unique dominant value in TU-games, but the axiomatic approach helps us evaluate the relative merits of values that have been proposed. An axiom (or a property) of a value is the mathematical expression of the intuition we have about how a value should behave in certain situations. In this section, we review some properties and axiomatizations of values from the literature.

[3] Recently, Li et al. (2020) studied a comparison between the PD value and the PANSC value in terms of optimizing satisfaction criteria and associated consistency.

1.3.1 Basic axioms

Players $i, j \in N$, $i \neq j$, are *symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. For $(N, v), (N, w) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, and $a, b \in \mathbb{R}$, the game $(N, av + bw) \in \mathcal{C}$ is defined by $(av + bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$. A *permutation* of N is a bijection $\pi : N \to N$ where $\pi(i) = k$ indicates that player i has the kth position. We denote $\Pi(N)$ as the set of all the $n!$ permutations of N.

The following properties are defined on $\mathcal{C} \subseteq \mathcal{G}$, which can be any class of games, depending on the specifications of games at work. Other properties mentioned in Section 1.3 can be defined for subclasses in an obvious way.

- **Efficiency**. For all $(N, v) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, it holds that $\sum_{i \in N} \psi_i(N, v) = v(N)$.
- **Symmetry**. For all $(N, v) \in \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{G}$, such that $i, j \in N$ are symmetric in (N, v), it holds that $\psi_i(N, v) = \psi_j(N, v)$.
- **Additivity**. For all $(N, v), (N, w) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, such that $(N, v + w) \in \mathcal{C}$, it holds that $\psi(N, v + w) = \psi(N, v) + \psi(N, w)$.
- **Linearity**. For all $(N, v), (N, w) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, and $a, b \in \mathbb{R}$ such that $(N, av + bw) \in \mathcal{C}$, it holds that $\psi(N, av + bw) = a\psi(N, v) + b\psi(N, w)$.
- **Anonymity**. For all $(N, v) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, and all permutations $\pi : N \to N$ such that $(N, \pi v) \in \mathcal{C}$, it holds that $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v)$ for all $i \in N$.

Efficiency states that all players together should totally allocate the worth of the grand coalition.

Symmetry states that if two players contribute the same amount to each coalition including neither of them, their payoffs are equal.

Linearity states that when taking a linear combination of two games, the payoff vector equals the corresponding linear combination of the payoff vectors of the two separate games. If $a = b = 1$, linearity reduces to additivity.

Anonymity states that the identity of the players does not affect payoffs of the players.

It is useful to evaluate a payoff vector for a class of games in relation to the corresponding payoff vector for its subgames. Some axioms describe the situations that only one payoff vector seems natural for games with a trivial structure.

- **Inessential game property**. For every additive game $(N, v) \in \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{G}$, it holds that $\psi_i(N, v) = v(\{i\})$ for all $i \in N$.
- **Null game property**. For the null game (N, v), it holds that $\psi_i(N, v) = 0$ for all $i \in N$.

The inessential game property states that if no additional contribution can be made by cooperation among players, all players get their stand-alone worth.

The null game property states that if the worths of all coalitions are zero, all players get zero.

It is also useful to evaluate a payoff vector for two related games that are almost the same.

- **Continuity**. For all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, such that $\lim_{k \to \infty} (N, w_k) = (N, v)$, it holds that $\lim_{k \to \infty} \psi(N, w_k) = \psi(N, v)$.

Continuity implies that if two games are almost the same, then their payoff vectors are almost the same.

To characterize the proportional Shapley value, Béal et al. (2018) and Besner (2019) respectively proposed weak linearity and weak additivity on $\mathcal{G}^N_{nz}$ by considering two games in which the stand-alone worths of all players are in the same proportion to each other.

- **Weak linearity** (Béal et al., 2018). For all $(N, v), (N, w) \in \mathcal{G}^N_{nz}$ and all $a \in \mathbb{R}$ such that $(N, av + w) \in \mathcal{G}^N_{nz}$ and there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, it holds that $\psi(N, av + w) = a\psi(N, v) + \psi(N, w)$.

- **Weak additivity** (Besner, 2019). For all $(N, v), (N, w) \in \mathcal{G}^N_{nz}$ such that $(N, v + w) \in \mathcal{G}^N_{nz}$ and there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, it holds that $\psi(N, v + w) = \psi(N, v) + \psi(N, w)$.

The condition that there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$ can be written as $v(\{i\}w(\{j\}) = v(\{j\})w(\{j\})$ for all $i, j \in N$.

Weak linearity states that when taking a linear combination of two games, where the ratio between the stand-alone worths is the same in both games, the payoff allocation equals the corresponding linear combination of the payoff vectors of the two separate games. If $a = 1$, weak linearity reduces to weak additivity. These axioms will be recalled in Chapters 2, 4 and 6, and also used in Chapter 5.

1.3.2 Null, nullifying, dummy, and dummifying

There are situations in which how much should be assigned to a player seems unequivocal.

We review notions of some special players in TU-games. Player $i \in N$ is a *null player* in game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. Player $i \in N$ is a *nullifying player* in game (N, v) if $v(S) = 0$ for all $S \subseteq N$ with $i \in S$. Player $i \in N$ is a *dummy player* in game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Player $i \in N$ is a *dummifying player* or a *separator* in game (N, v) if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ with $i \in S$.

Notice the difference between a dummy and dummifying player: a dummy player adds its own stand-alone worth when it joins any coalition, while a dummifying player entering a coalition results in the worth of that coalition becoming equal to the sum of the stand-alone worths of the players in that coalition.

Again, the following properties, as well as other properties in Section 1.3, are defined for subclasses in an obvious way like the axioms stated in Subsection 1.3.1.

- **Null player property**. For all $(N, v) \in \mathcal{G}^N$ such that $i \in N$ is a null player in (N, v), it holds that $\psi_i(N, v) = 0$.
- **Nullifying player property** (van den Brink, 2007). For all $(N, v) \in \mathcal{G}^N$ such that $i \in N$ is a nullifying player in (N, v), it holds that $\psi_i(N, v) = 0$.
- **Dummy player property**. For all $(N, v) \in \mathcal{G}^N$ such that $i \in N$ is a dummy player in (N, v), it holds that $\psi_i(N, v) = v(\{i\})$.
- **Dummifying player property** (Casajus and Huettner, 2014a). For all $(N, v) \in \mathcal{G}^N$ such that $i \in N$ is a dummifying player in (N, v), it holds that $\psi_i(N, v) = v(\{i\})$.

The null player axiom states that if a player's contribution to each coalition is zero, she gets zero.

The nullifying player axiom states that if a player is such that the worth of each coalition containing her is zero, she gets zero.

The dummy player axiom states that if a player's contribution to each coalition is equal to her own stand-alone worth, she gets her own stand-alone worth.

The dummifying player property states that a dummifying player just earns its own stand-alone worth.

Various axiomatizations of the Shapley value have been given in the literature. One of the most famous uses efficiency, symmetry, additivity, and the null player property, see Shapley (1953a).

Theorem 1.1 (Shapley, 1953a)**.** *The Shapley value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, additivity, and the null player property.*

Interestingly, using the nullifying player property instead of the null player property, van den Brink (2007) characterizes the ED value. Later, using the dummifying player property instead of the null player property, Casajus and Huettner (2014a) characterizes the ESD value.

Theorem 1.2 (van den Brink, 2007)**.** *The ED value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, additivity, and the nullifying player property.*

Theorem 1.3 (Casajus and Huettner, 2014a)**.** *The ESD value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, additivity, and the dummifying player property.*

1.3.3 Monotonicity

In the axiomatic formulation of TU-games, monotonicity is an important characteristic of viable and stable solutions. A monotonicity axiom expresses the requirement that a player's payoff should move in a particular direction if a TU-game changes in certain ways that are 'advantageous' for this player. It may be felt that a player is entitled to at least as much as what she was assigned initially. Due to the richness of the description of a game, these axioms come in a great variety of forms, such as these monotonicity axioms listed in this section.

- **Strong monotonicity** (Young, 1985). For all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) - v(S\backslash\{i\}) \geq w(S) - w(S\backslash\{i\})$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(N, v) = \psi_i(N, w)$.

Strong monotonicity implies comparing the payoffs attributed to a player by a value in certain games with the same player set. If in one game the player contributes more, i.e., has a higher marginal contribution, to any coalition than in the other, then the amount attributed to her by the value in the former should not be smaller than in the latter game. Young characterized the Shapley value with strong monotonicity.

Theorem 1.4 (Young, 1985)**.** *The Shapley value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, and strong monotonicity.*

Under efficiency and symmetry, van den Brink (2007) proved that coalitional standard equivalence and coalitional monotonicity characterize the ED value.

- **Coalitional standard equivalence** (van den Brink, 2007). For all $(N, v), (N, w) \in \mathcal{G}^N$ such that $i \in N$ is a nullifying player in (N, w), it holds that $\psi_i(N, v + w) = \psi_i(N, v)$.
- **Coalitional monotonicity** (van den Brink, 2007). For all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(N, v) \geq \psi_i(N, w)$.

Coalitional standard equivalence states that the payoff of a player remains unchanged if we add a game in which this player is a nullifying player.

Coalitional monotonicity states that the payoff of a player should not decrease whenever the worth of every coalition containing this player weakly increases.

Theorem 1.5 (van den Brink, 2007)**.** *The ED value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, and coalitional standard equivalence.*

Theorem 1.6 (van den Brink, 2007)**.** *The ED value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, and coalitional monotonicity.*

Following van den Brink (2007), Casajus and Huettner (2014a) proposed coalitional surplus equivalence and coalitional surplus monotonicity, and proved that either of them in addition to efficiency and symmetry characterize the ESD value.

- **Coalitional surplus equivalence** (Casajus and Huettner, 2014a). For all (N, v), $(N, w) \in \mathcal{G}^N$ such that $i \in N$ is a dummifying player in (N, w), it holds that $\psi_i(N, v + w) = \psi_i(N, v) + w(\{i\})$.

- **Coalitional surplus monotonicity** (Casajus and Huettner, 2014a). For all (N, v), $(N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) - \sum_{j \in S} v(\{j\}) \geq w(S) - \sum_{j \in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(N, v) - v(\{i\}) \geq \psi_i(N, w) - w(\{i\})$.

Coalitional surplus equivalence states that the payoff of a player increases by her stand-alone worth if we add a game in which this player is a dummifying player.

Coalitional surplus monotonicity states that if two games in which the surplus of every coalition a player belongs to (measured by the worth of the coalition minus the sum of the stand-alone worths of its players) weakly increases, then the relative payoff of this player (being the difference between the payoff and the stand-alone worth) should not decrease.

Theorem 1.7 (Casajus and Huettner, 2014a). *The ESD value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, and coalitional surplus equivalence.*

Theorem 1.8 (Casajus and Huettner, 2014a). *The ESD value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, and coalitional surplus monotonicity.*

It is shown in van den Brink and Funaki (2009) that the ED value is the unique value belonging to the family of efficient, symmetric and linear values that satisfies nonnegativity.

- **Nonnegativity**. For all $(N, v) \in \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{G}$, with $v(N) \geq 0$, it holds that $\psi_i(N, v) \geq 0$ for all $i \in N$.

Theorem 1.9 (van den Brink and Funaki, 2009). *The ED value is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, linearity, and nonnegativity.*

Next, we present other monotonicity axioms that will be used in this thesis by modifying the domain into other subclasses.

- **Grand coalition monotonicity** (Casajus and Huettner, 2014b). For all (N, v), $(N, w) \in \mathcal{G}^N$ with $v(N) \geq w(N)$, it holds that $\psi_i(N, v) \geq \psi_i(N, w)$ for all $i \in N$.

- **Id+sur monotonicity** (Yokote and Funaki, 2017). For all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(\{i\}) \geq w(\{i\})$ and $v(N) - \sum_{j \in N} v(\{j\}) \geq w(N) - \sum_{j \in N} w(\{j\})$, it holds that $\psi_i(N, v) \geq \psi_i(N, w)$.

- **Superadditive monotonicity** (Ferrières, 2017). For every superadditive and monotone game $(N, v) \in \mathcal{G}^N$, it holds that $\psi_i(N, v) \geq 0$ for all $i \in N$.

- **Desirability** (Maschler and Peleg, 1966). For all $(N, v) \in \mathcal{G}^N$ and $i, j \in N$ such that $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, it holds that $\psi_i(N, v) \geq \psi_j(N, v)$.

Grand coalition monotonicity states that if two games in which the worth of the grand coalition weakly increases, then the payoffs of all players should not decrease.

Id+sur monotonicity states that if two games in which the stand-alone worth of a player and the surplus of the grand coalition weakly increases, then the payoff of this player should not decrease.

Superadditive monotonicity states that a player's payoff is nonnegative in a superadditive and monotone TU-game.

Desirability states that if i's contributions are not less than j's contributions, then i should receive at least j's payoff.

1.3.4 Relational contributions

The relational contributions of two players also play a crucial role in the intuition of fairness. The relational contribution refers to some specific changes or some invariance principle on the payoffs according to particular modifications of the game. One of the most well-known of such properties is balanced contributions introduced by Myerson (1980), who first considers the change in payoff of a player when other player leaves the game. We begin this subsection by recalling the balanced contributions axiom.

Given $(N, v) \in \mathcal{G}^N$ and $i \in N$, the game $(N\backslash\{i\}, v_{N\backslash\{i\}})$ on player set $N\backslash\{i\}$ is defined by $v_{N\backslash\{i\}}(S) = v(S)$ for all $S \subseteq N\backslash\{i\}$. For simplicity, $(N\backslash\{i\}, v_{N\backslash\{i\}})$ is written as $(N\backslash\{i\}, v)$.

Suppose that players agree to use a value ψ whenever the grand coalition forms. Then $\psi_i(N, v) - \psi_i(N\backslash\{j\}, v)$ is the amount player i gains or loses when N is already formed and player j resigns. The balanced contributions axiom requires that the amounts that each player would gain or lose by the other's withdrawal from the coalition should be equal.

- **Balanced contributions** (Myerson, 1980). For all $(N, v) \in \mathcal{G}^N$ and $i, j \in N$, it holds that $\psi_i(N, v) - \psi_i(N\backslash\{j\}, v) = \psi_j(N, v) - \psi_j(N\backslash\{i\}, v)$.

Theorem 1.10 (Myerson, 1980)**.** *The Shapley value is the unique value on $\mathcal{G}^N$ that satisfies efficiency and balanced contributions.*

Instead of considering a variable player set, as balanced contributions does, several related properties are formulated on a fixed player set. In van den Brink and Funaki (2009), it is supposed that player $h \in N$ becomes a veto player in game (N, v), i.e. instead of characteristic function v, and consider the characteristic function v^h_{vo} given by

$$v^h_{vo}(S) = \begin{cases} v(S) & \text{if } h \in S, S \subseteq N, \\ 0 & \text{otherwise.} \end{cases}$$

Based on this operation, which is called *veto-ification* by Ferrières (2017), van den Brink and Funaki (2009) suggest the *veto equal loss property* by requiring that veto-ification yields the same change in payoff for the other players when a game (N, v) is zero-normalized, i.e. $v(i) = 0$ for all $i \in N$.

- **Veto equal loss property**. For every zero-normalized game (N, v) with $|N| \geq 3$, all $h \in N$ and all $i, j \in N \backslash \{h\}$, it holds that

$$\psi_i(N, v) - \psi_i(N, v^h_{vo}) = \psi_j(N, v) - \psi_j(N, v^h_{vo}).$$

Considering the effect on a player becoming a null player in a game, Ferrières (2017) and Kongo (2018) independently suggested the nullified equal loss property in axiomatizing the ED value, the ESD value, and the classes of their affine and convex combinations. Given $(N, v) \in \mathcal{G}^N$ and $h \in N$, (N, v^h_0) denotes player h becoming a null player in (N, v), i.e. the characteristic function v^h_0 is given by

$$v^h_0(S) = \begin{cases} v(S \backslash \{h\}) & \text{if } h \in S, S \subseteq N, \\ v(S) & \text{otherwise.} \end{cases}$$

- **Nullified equal loss property**. For all $(N, v) \in \mathcal{G}^N$ with $|N| \geq 3$, all $h \in N$ and all $i, j \in N \backslash \{h\}$, it holds that

$$\psi_i(N, v) - \psi_i(N, v^h_0) = \psi_j(N, v) - \psi_j(N, v^h_0).$$

Theorem 1.11 (Ferrières, 2017). *Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N$ satisfies efficiency, the nullified equal loss property, linearity, and symmetry if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1 - \beta) ED$.*

Ferrières (2017) and Kongo (2018) also provide axiomatic results by employing the monotonicity axioms, see Subsection 1.3.3.

Theorem 1.12 (Ferrières, 2017). *Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N$ satisfies efficiency, the nullified equal loss property, additivity, desirability, and superadditive monotonicity if and only if there is $\beta \in [0, 1]$ such that $\psi = \beta ESD + (1 - \beta) ED$.*

Theorem 1.13 (Kongo, 2018). *Let $|N| \geq 3$. Let ψ be a value on $\mathcal{G}^N$ that satisfies efficiency, the nullified equal loss property, and the null game property. Then,*

(i) ψ satisfies grand coalition monotonicity if and only if $\psi = ED$.

(ii) ψ satisfies Id+sur monotonicity if and only if $\psi = ESD$.

Similarly, Béal et al. (2018) offer a characterization of the proportional Shapley value on $\mathcal{G}^N_{nz}$ by considering the effect on a player of becoming a dummy player in a game.

- **Proportional balanced contributions under dummification** (Béal et al., 2018). For all $(N,v) \in \mathcal{G}^N_{nz}$ and $i,j \in N$, it holds that

$$\frac{\psi_i(N,v) - \psi_i(N,v^j_d)}{v(\{i\})} = \frac{\psi_j(N,v) - \psi_j(N,v^i_d)}{v(\{j\})},$$

where $(N,v^i_d) \in \mathcal{G}^N_{nz}$ is the game in which i is *dummified*: $v^i_d(S) = v(S\setminus\{i\}) + v(\{i\})$ for all $S \subseteq N$ with $i \in S$, and $v^i_d(S) = v(S)$ for all $S \not\ni i$.

Theorem 1.14 (Béal et al., 2018)**.** *The proportional Shapley value is the unique value on $\mathcal{G}^N_{nz}$ that satisfies efficiency, proportional balanced contributions under dummification, and the inessential game property.*

1.3.5 Consistency

We now turn to notions of consistency. Consistency principles, first investigated for abstract models of cooperative game theory, have now been examined in the context of a great variety of resource allocation problems. Consistency of a value in TU-games is described as follows. Given a payoff vector chosen by the value for some initial game, and given a subgroup of players, a so-called *reduced game* among them is constructed from the initial game by giving the rest of the players payoffs according to the payoff vector. The value is *consistent* if it selects the same payoff allocation over the remaining players for the reduced game as initially.

In the literature, various values in TU-games are characterized by means of a consistency property in terms of the reduced games. Without going into detials, we mention that such values include, e.g., the Shapley value, the ED value, the ESD value, the EANSC value, the kernel (Davis and Maschler, 1965), the prekernel (Maschler et al., 1971), the nucleolus (Schmeidler, 1969), and the Core (Gillies, 1953). Most of the contributions of these values can be found in some surveys, see Driessen (1991), Thomson (1990), and Thomson (2011a).

In this thesis, we will characterize the PD value involving projection consistency used in Funaki and Yamato (2001), van den Brink and Funaki (2009), van den Brink et al. (2016), Calleja and Llerena (2017), and Calleja and Llerena (2019). In van den Brink and Funaki (2009) and van den Brink et al. (2016), projection consistency is employed in evaluating egalitarian values, such as the ED value, the ESD value, the EANSC value, and the class of combinations of them.

If a player $j \in N$ leaves game (N,v) with a certain payoff, then the *projection reduced game* is a game on the remaining player set that assigns to every proper subset of $N \setminus \{j\}$ its worth in the original game, and to coalition $N \setminus \{j\}$ assigns the worth $v(N)$ in (N,v) minus the payoff assigned to player j.

Definition 1.7. Given a game $(N,v) \in \mathcal{G}$ with $|N| \geq 2$, a player $j \in N$ and a payoff vector $x \in \mathbb{R}^N$, the *projection reduced game with respect to j and x* is the game

$(N\backslash\{j\}, v^x)$ given by

$$v^x(S) = \begin{cases} v(N) - x_j & \text{if } S = N\backslash\{j\}, \\ v(S) & \text{if } S \subset N\backslash\{j\}. \end{cases}$$

Projection consistency requires that the payoffs assigned to the remaining players in $N \setminus \{j\}$, after player j leaving the game with its payoff according to a value ψ, is the same in the reduced game as in the original game.

Definition 1.8. A value ψ satisfies *projection consistency* if for every game $(N, v) \in \mathcal{G}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N, v)$, it holds that $\psi_i(N\backslash\{j\}, v^x) = \psi_i(N, v)$ for all $i \in N\backslash\{j\}$.

This consistency principle will be recalled in Subsection 2.3.3, but the domain $\mathcal{G}$ is replaced by $\mathcal{G}_{nz}$ (and thus the definition should be slightly modified).

A sequence of the reduced games from an n-player game yields a two-player game, so the consistency principles are usually considered together with the properties only for two-player games. For instance, the Shapley value is characterized by a consistency principle and *standardness* in Hart and Mas-Colell (1989). *Standardness* assigns each player its stand-alone worth and allocates the surplus equally over all players for two-player games.

Definition 1.9 (Hart and Mas-Colell, 1989)**.** A value ψ satisfies *standardness* if for every game $(N, v) \in \mathcal{G}^N$ with $|N| = 2$, it holds that

$$\psi_i(N, v) = v(\{i\}) + \frac{1}{2}[v(N) - v(\{i\}) - v(\{j\})], \;\; \forall i, j \in N.$$

Similarly, *α-standardness for two-player games* assigns each player the fraction α of its stand-alone worth and allocates the surplus equally over all players for two-player games. This axiom is first introduced in Joosten (1996) for axiomatizing the α-egalitarian Shapley values. Later on, van den Brink and co-authors did much more with it. For example, van den Brink et al. (2013) combine it with Sobolev consistency to obtain the class of egalitarian Shapley values; van den Brink and Funaki (2009) and van den Brink et al. (2016) combine it with projection consistency to obtain the class of egalitarian values.

Definition 1.10 (Joosten, 1996)**.** A value ψ satisfies *α-standardness for two-player games* if for every game $(N, v) \in \mathcal{G}^N$ with $|N| = 2$, it holds that

$$\psi_i(N, v) = \alpha v(\{i\}) + \frac{1}{2}[v(N) - \alpha(v(\{i\}) + v(\{j\}))], \;\; \forall i, j \in N.$$

Both standardness and α-standardness for two-player games indicate an equality principle, whereas the following proportional standardness indicates a proportionality principle.

Proportional standardness, also known as "proportional for two person games" in Ortmann (2000), requires that in two-player games we allocate the worth of the grand coalition over the two players proportional to their stand-alone worths. This is equivalent to saying that every player in a two-player game earns its own stand-alone worth, and the remainder of the worth is shared proportionally based on their stand-alone worths.

Definition 1.11 (Ortmann, 2000)**.** A value ψ satisfies *proportional standardness* if for every game $(N, v) \in \mathcal{G}_{nz}$ with $|N| = 2$, it holds that

$$\psi_i(N, v) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})}[v(N) - v(\{i\}) - v(\{j\})], \ \ \forall i, j \in N.$$

It is known that the Shapley value (Shapley, 1953a) and the ESD value (Driessen and Funaki, 1991) satisfy standardness, α-egalitarian Shapley values (Joosten, 1996) and the ED value (axiomatized in van den Brink (2007)) satisfy egalitarian standardness, and various proportional values, such as the proportional value (Ortmann, 2000), the proportional Shapley value (Béal et al., 2018; Besner, 2019), the proper Shapley values (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015), and the proportional Harsanyi solution (Besner, 2020) satisfy proportional standardness. Without going into details, we only recalled the following result.

Theorem 1.15 (van den Brink et al., 2016)**.** *Let $\alpha \in [0, 1]$. A value ψ on $\mathcal{G}$ satisfies projection consistency and α-standardness for two-player games if and only if $\psi = \alpha ESD + (1 - \alpha)ED$.*

For ease of the reader, we will repeat the relevant definitions in the following chapters.

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Chapter 2

Axiomatizations of the Proportional Division Value

2.1 Introduction

Proportionality is an often applied equity principle in allocation problems. The idea of proportionality can be traced at least as far back as Aristotle's celebrated maxim, "*Equals should be treated equally, unequals unequally, in proportion to relevant similarities and differences*" from Nicomachean Ethics. In this chapter, which is based on Zou et al. (2021), we consider the proportionality principle in the context of TU-games.

With a natural proportionality consideration, the *proportional rule* (Moriarity, 1975; Banker, 1981) allocates the worth of the grand coalition in proportion to the stand-alone worths of its members. In the thesis, we call this the *proportional division value*, shortly denoted by the PD value, in order to distinguish it from the proportional rule in bankruptcy problems, claims problems, bargaining problems, insurance, law and so on.[1]

Moulin (1987) characterizes the PD value for joint venture games, being a class of TU-games where intermediate coalitions are *inessential*, in the sense that the worth of every proper subset of the full player set equals the sum of the worths of its stand-alone coalitions. These are the *quasi-additive games* in Carreras and Owen (2013), where the PD value is discussed by comparing it with the Shapley value (Shapley, 1953a). Banker (1981) considers the situation that the worth of a coalition is a non-negative strictly increasing function with respect to the sum of the worths of its members. However, for more general TU-games, since the proportionality principle is not obvious, as far as we know, an axiomatic characterization of the PD value is still missing.

In this chapter, we axiomatize the PD value on the domain of TU-games in which the worths of all singleton coalitions are nonzero and have the same sign. This restrictive class of TU-games is considered in Béal et al. (2018) who also provide many

[1] For other proportional solutions, we refer to the proportional value (Ortmann, 2000; Khmelnitskaya and Driessen, 2003; Kamijo and Kongo, 2015), the proper Shapley values (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015), the proportional Shapley value (Béal et al., 2018; Besner, 2019), and the proportional Harsanyi solution (Besner, 2020).

applications, such as airport games (Littlechild and Owen, 1973), highway cost sharing problems (Kuipers et al., 2013), data sharing games (Dehez and Tellone, 2013). We focus on some intuitive fairness criteria that are widely used in the theory for TU-games, including *equal treatment of equals* (also known as *symmetry*), *monotonicity*, and *consistency*.

First, we introduce a proportionality principle called *proportional-balanced treatment* in TU-games, which is a strengthening of Shapley's symmetry axiom. It states that the payoffs to two players whose contribution to every nonempty coalition not containing them is the same (we call this weak symmetric players), are proportional to their stand-alone worths. It well captures the principles of 'equal treatment of equals' and 'unequal treatment of unequals'. Besner (2019) gives a similar axiom for the proportional Shapley value. Interestingly, proportional-balanced treatment together with efficiency and weak linearity as introduced in Béal et al. (2018), give a family of values that have a formula similar as the family of efficient linear and symmetric values (ELS values for short) introduced in Ruiz et al. (1998), but where the role of equal division is replaced by proportional division. While the Shapley value is the only ELS value that satisfies the dummy player property, we reveal that there is *no* value belonging to our family that satisfies the dummy player property. Instead, we adopt the dummifying player property introduced in Casajus and Huettner (2014a), and obtain a characterization of the PD value.

Second, we provide characterizations of the PD value by applying weaker versions of well-known monotonicity axioms. A monotonicity axiom states that the payoff of a player should not decrease if a TU-game changes in certain ways that are 'advantageous' for this player. We introduce three such monotonicity axioms that are a relaxation of three existing axioms, by adding restrictions on the stand-alone worths of the players.[2] The three existing axioms are coalitional monotonicity due to van den Brink (2007), and coalitional surplus equivalence and coalitional surplus monotonicity, both axioms due to Casajus and Huettner (2014a). Not surprisingly, any of our monotonicity axioms together with efficiency and symmetry cannot characterize a unique value. However, replacing symmetry by proportional-balanced treatment and any of our monotonicity axioms, characterizes the PD value.

For a variable player set, we provide an axiomatization of the PD value using proportional standardness and the well-known projection consistency due to Funaki and Yamato (2001). Proportional standardness requires to apply proportional division for two-player games, and is used in Ortmann (2000), Khmelnitskaya and Driessen (2003), van den Brink and Funaki (2009), and Huettner (2015). Like other standardness axioms, proportional standardness is rather strong since it sets the payoff alloction for two-player games. Therefore, we conclude with characterizing proportional standardness on the class of two-player games.

[2]This modification is similar in spirit to parameterized monotonicity introduced in Yokote and Funaki (2017).

This chapter is organized as follows. Section 2.2 recalls basic definitions and notation. Section 2.3 contains four subsections and deals with characterizations of the PD value. In Subsection 2.3.1, we introduce proportional-balanced treatment and provide some results including an axiomatic charaterization of the PD value. In Subsection 2.3.2, we offer three axiomatic charaterizations using some monotonicity axioms. In Subsection 2.3.3, we give an axiomatic charaterization on variable player sets by employing projection consistency and proportional standardness. In Subsection 2.3.4, we characterize proportional standardness for two-player games. The logical independence among the axioms in the provided characterizations is analyzed in Section 2.4. The proofs are provided in Section 2.5. Section 2.6 concludes.

2.2 Definitions and notation

We recall some definitions and notation. As mentioned in the introduction, we restrict our discussion to the class $\mathcal{G}^N_{nz}$ that consists of all individually positive and individually negative games on given player set N, i.e., $\mathcal{G}^N_{nz} = \{(N,v) \in \mathcal{G}^N \mid v(\{i\}) > 0 \text{ for all } i \in N, \text{ or } v(\{i\}) < 0 \text{ for all } i \in N\}$.

The *proportional division (PD) value* on $\mathcal{G}^N_{nz}$ is given by

$$PD_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) \tag{2.1}$$

for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

The following properties of values, stated in Chapter 1 for arbitrary subclasses of games, will be considered in this chapter.

- **Efficiency**. For all $(N,v) \in \mathcal{G}^N_{nz}$, it holds that $\sum_{i\in N} \psi_i(N,v) = v(N)$.
- **Symmetry**. For all $(N,v) \in \mathcal{G}^N_{nz}$ such that $i,j \in N$ are symmetric in (N,v), it holds that $\psi_i(N,v) = \psi_j(N,v)$.
- **Dummy player property**. For all $(N,v) \in \mathcal{G}^N_{nz}$ such that $i \in N$ is a dummy player in (N,v), it holds that $\psi_i(N,v) = v(\{i\})$.
- **Dummifying player property**. For all $(N,v) \in \mathcal{G}^N_{nz}$ such that $i \in N$ is a dummifying player in (N,v), it holds that $\psi_i(N,v) = v(\{i\})$.
- **Weak linearity**. For all $a \in \mathbb{R}$, and all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ such that there exists $c \in \mathbb{R}_+$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, if $(N, av + w) \in \mathcal{G}^N_{nz}$, then $\psi(N, av + w) = a\psi(N,v) + \psi(N,w)$.

The first four axioms are classical, except that they are defined on subclass $\mathcal{G}^N_{nz}$. Weak linearity, proposed by Béal et al. (2018), states that when taking a linear combination of two games, where the ratio between the stand-alone worths is the same in both games, the payoff allocation equals the corresponding linear combination of the payoff vectors of the two separate games. This axiom is a weak version of the axiom of linearity as proposed by Shapley (1953a). If $a = 1$, then weak linearity reduces to **weak additivity**, which is introduced and studied in Besner (2019).

2.3 Axiomatic characterizations

This section aims to provide characterizations of the PD value for TU-games.

2.3.1 Proportionality principle

In this subsection, we introduce a new axiom, called proportional-balanced treatment, and characterize the PD value.

Definition 2.1. Players $i, j \in N$, $i \neq j$, are *weak symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, $S \neq \emptyset$.

Two players being weak symmetric still allows them to have a different stand-alone worth, but their contribution to any nonempty coalition including neither of them should be equal. Notice that in two-player games, both players are always weak symmetric. We now introduce a proportionality property, comparable to symmetry, which says that the payoffs to two weak symmetric players are in the same proportion as their stand-alone worths. This axiom can be considered as a strengthening of Shapley's symmetry axiom since it implies that any two symmetric players in any game should earn the same payoff.

- **Proportional-balanced treatment**. For all $(N, v) \in \mathcal{G}_{nz}^N$ such that $i, j \in N$ are weak symmetric players in (N, v), it holds that $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$.

Next, we exactly characterize the class of values on $\mathcal{G}_{nz}^N$ that satisfies efficiency, weak linearity, and proportional-balanced treatment.

Theorem 2.1. *A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, weak linearity, and proportional-balanced treatment if and only if for all $(N, v) \in \mathcal{G}_{nz}^N$ and all $i \in N$,*

$$\begin{aligned} &\psi_i(N, v) \\ &= \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) + v(\{i\}) \Bigg[\sum_{\substack{S: i \in S \neq N \\ |S| \geq 2}} \frac{\lambda_S}{\sum_{j \in S} v(\{j\})} v(S) - \sum_{\substack{S: i \notin S \\ |S| \geq 2}} \frac{\lambda_S}{\sum_{j \in N \backslash S} v(\{j\})} v(S) \Bigg], \end{aligned} \tag{2.2}$$

where for each $S \subset N$ with $|S| \geq 2$, λ_S is a real number such that

$$\frac{\lambda_S}{\sum_{j \in S} v(\{j\}) \sum_{j \in N \backslash S} v(\{j\})} = \frac{\lambda_T}{\sum_{j \in T} v(\{j\}) \sum_{j \in N \backslash T} v(\{j\})}, \quad \textit{if } |S| = |T|. \tag{2.3}$$

The proof of Theorem 2.1 and of all other results in this chapter can be found in Section 2.5. The proof of Theorem 2.1 uses the following proposition.

Proposition 2.1. *Let $N \in \mathcal{N}$ with $|N| = 2$. The PD value is the unique value on $\mathcal{G}_{nz}^N$ satisfying efficiency and proportional-balanced treatment.*

The values characterized in Theorem 2.1 can be seen as modifications of the PD value, where to every game they first apply the PD value and then make a 'correction' that is based on the stand-alone worth of a player and the difference between weighted sums of the worths of all other coalitions with and without this player. The weights depend on all stand-alone worths. In this sense, (2.2) bears some similarity with the family of efficient, linear and symmetric (ELS) values (Lemma 9, Ruiz et al., 1998) which can be written as:

$$\psi_i(N,v) = \frac{v(N)}{|N|} + \sum_{S:i\in S\neq N} \frac{\rho_s}{|S|} v(S) - \sum_{S:i\notin S} \frac{\rho_s}{|N|-|S|} v(S), \tag{2.4}$$

where ρ_s, $s \in \{1,2,\ldots,n-1\}$, is a real number. The ELS values can be seen as first applying equal division and then make a correction based on a weighted sum of differences between worths of coalitions with and without a player. Specifically, if $v(\{i\}) = v(\{j\})$ for all $i,j \in N$, (2.2) coincides with the above equation.

Remark 2.1. Note that (2.3) indicates that all coefficients of coalitions of the same size are uniquely determined as soon as any one of them is given. For computational convenience, given $\{\lambda_S \in \mathbb{R} \mid S \subset N, |S| \geq 2\}$, denoting $\lambda_s = \frac{\lambda_S}{\sum_{j\in S} v(\{j\}) \sum_{j\in N\setminus S} v(\{j\})}$, (2.2) can be rewritten as

$$\begin{aligned} &\psi_i(N,v) \\ &= \frac{v(\{i\})v(N)}{\sum_{j\in N} v(\{j\})} + v(\{i\}) \left[\sum_{S:i\in S\neq N} \sum_{j\in N\setminus S} v(\{j\})\lambda_s v(S) - \sum_{S:i\notin S} \sum_{j\in S} v(\{j\})\lambda_s v(S) \right], \end{aligned} \tag{2.5}$$

where $\lambda_1 = 0$, and λ_s, $s \in \{2,\ldots,n-1\}$, is a function with respect to λ_S and all stand-alone worths. Since λ_s might be different for different games which stand-alone worths are different, (2.5) cannot be directly used to verify weak linearity.

Remark 2.2. A family of values derived from the family of ELS values given by (2.4) with $\rho_1 = 0$, satisfies proportional-balanced treatment as follows. For any ELS value ψ' given by (2.4) with $\rho_1 = 0$, the value ψ defined by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} \psi'_i(N,v) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} \left[v(N) - \sum_{h\in N} \frac{v(\{h\})\psi'_h(N,v)}{\sum_{k\in N} v(\{k\})} \right] \tag{2.6}$$

satisfies proportional-balanced treatment, and also efficiency and weak linearity. This value can be viewed as a multiplicative normalization of an ELS value.

A next question is whether the class of values characterized in Theorem 2.1 contains a value that satisfies the dummy player property. It turns out that, for games with at least three players, the dummy player property is incompatible with the three axioms in Theorem 2.1.

Theorem 2.2. *Let $|N| \geq 3$. There is no value on $\mathcal{G}_{nz}^N$ satisfying efficiency, weak linearity, proportional-balanced treatment, and the dummy player property.*

Notice that for $|N| = 2$, the PD value satisfies these axioms.

Since the PD value satisfies efficiency, weak linearity, and proportional-balanced treatment on $\mathcal{G}_{nz}^N$, it belongs to the class of values characterized in Theorem 2.1. In fact, it is the value corresponding to $\lambda_S = 0$ for all $S \subseteq N$. As it turns out, replacing the dummy player property in Theorem 2.2 by the dummifying player property, characterizes the PD value (also holds for two-player games).

Theorem 2.3. *The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, weak linearity, proportional-balanced treatment, and the dummifying player property.*

Logical independence of the axioms in this theorem, as well as other theorems in this chapter, is shown in Section 2.4.

Remark 2.3. If the domain is relational to the class containing all *individually equal games* (i.e., $(N, v) \in \mathcal{G}_{nz}^N$ such that $v(\{i\}) = v(\{j\})$ for all $i, j \in N$), then weak linearity and proportional-balanced treatment reduce to linearity and symmetry (since for any game in this class, all stand-alone worths are the same), respectively. Denoting this class of games by $\mathcal{G}_e^N$, in contrast to Theorem 2.2 and Theorem 2.3, one can obtain the following results: (i) The Shapley value is the unique value on $\mathcal{G}_e^N$ that satisfies efficiency, additivity, symmetry, and the dummy player property; (ii) The equal division (ED) value is the unique value on $\mathcal{G}_e^N$ that satisfies efficiency, additivity, symmetry, and the dummifying player property.

Remark 2.4. Besner (2019) characterizes the proportional Shapley value by employing a proportionality axiom, which says $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for all $(N, v) \in \mathcal{G}_{nz}^N$ and $i, j \in N$ such that $v(S \cup \{k\}) = v(S) + v(\{k\})$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$. Clearly, this axiom focuses on a pair of weakly dependent players, whereas proportional-balanced treatment considers weak symmetric players.

We conclude this section by comparing our results with the main results in Casajus and Huettner (2014a), which show that on the domain of TU-games $\mathcal{G}^N$, the equal surplus division (ESD) value treats dummifying players in the same way as the Shapley value handles dummy players. Restricting ourselves to the subclass $\mathcal{G}_{nz}^N$, notice that the PD value is a variation of both the ED value and the ESD value since $PD_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}[v(N) - \sum_{j \in N} v(\{j\})]$. Interestingly, Theorem 2.3 gives a characterization of the PD value using the dummifying player property, whereas, for $|N| \geq 3$, using the dummy player property instead of the dummifying player property leads to an impossibility, as shown in Theorem 2.2.

2.3.2 Monotonicity

In this subsection, we present axiomatic characterizations of the PD value by imposing three appropriate monotonicity axioms being weaker versions of classical monotonicity axioms in the literature.

- **Weak coalitional surplus equivalence**[3]. For all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ with $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ being a dummifying player in (N,w), it holds that $\psi_i(N,v+w) = \psi_i(N,v) + w(\{i\})$.

- **Weak coalitional surplus monotonicity**. For all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ with $w(\{j\}) = cv(\{j\})$ for all $j \in N$ and $c \in \mathbb{R}_+$, and $i \in N$ such that $v(S) - \sum_{j\in S} v(\{j\}) \geq w(S) - \sum_{j\in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(N,v) - v(\{i\}) \geq \psi_i(N,w) - w(\{i\})$.

- **Weak coalitional monotonicity**. For all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ with $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(N,v) \geq \psi_i(N,w)$.

Weak coalitional surplus equivalence states that the payoff of a player increases by her stand-alone worth if we add a game in which this player is a dummifying player and each stand-alone worth is the same as that of the original game.

Weak coalitional surplus monotonicity states that if two games in which the stand-alone worths of all players are in the same proportion to each other and the surplus of every coalition a player belongs to (measured by the worth of the coalition minus the sum of the stand-alone worths of its players) weakly increases, then the relative payoff of this player (being the difference between the payoff and the stand-alone worth) should not decrease.

Weak coalitional monotonicity states that the payoff of a player should not decrease whenever the worth of every coalition containing this player weakly increases, while the worth of every singleton coalition remains unchanged.

Weak coalitional surplus equivalence (respectively, weak coalitional surplus monotonicity) is a weak version of coalitional surplus equivalence (respectively, coalitional surplus monotonicity) as defined in Casajus and Huettner (2014a). Weak coalitional monotonicity is stronger than coalitional monotonicity[4] as defined in Shubik (1962), while it is weaker than coalitional monotonicity as defined in van den Brink (2007).[5] Not surprisingly, any of our monotonicity axioms together with efficiency and symmetry cannot characterize a unique value. Outstandingly, replacing symmetry by proportional-balanced treatment and keeping efficiency, we derive that any of our monotonicity axioms characterizes the PD value.

Notice that weak coalitional monotonicity is a specific case of weak coalitional surplus monotonicity taking $c = 1$. In addition, weak coalitional surplus monotonicity implies weak coalitional surplus equivalence.

[3]Weak coalitional surplus equivalence is a monotonicity principle since it is implied by weak coalitional surplus monotonicity with $c = 2$. See Lemma 2.1.

[4]A value ψ satisfies Shubik's version of *coalitional monotonicity* if $\psi_i(N,v) \geq \psi_i(N,w)$ for all $(N,v),(N,w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, and $v(S) = w(S)$ for all $S \subseteq N\backslash\{i\}$.

[5]Under efficiency and symmetry, coalitional monotonicity characterizes the equal division value in van den Brink (2007), and either coalitional surplus equivalence or coalitional surplus monotonicity characterizes the equal surplus division value in Casajus and Huettner (2014a).

Lemma 2.1. *On $\mathcal{G}_{nz}^N$, weak coalitional surplus monotonicity implies weak coalitional surplus equivalence.*

Considering weak coalitional surplus equivalence and weak coalitional surplus monotonicity, the PD value is characterized by either one of these axioms in addition to efficiency and proportional-balanced treatment.

Theorem 2.4. *(i) The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus equivalence.*

(ii) The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus monotonicity.

The next lemma shows a logical implication between the axioms in Theorem 2.3 and Theorem 2.4(i), which implies that weak linearity in Theorem 2.3 can be weakened as weak additivity.

Lemma 2.2. *Weak additivity and the dummifying player property together imply weak coalitional surplus equivalence.*

It is easy to verify that the PD value satisfies weak coalitional monotonicity. Interestingly, the PD value is characterized by replacing weak coalitional surplus monotonicity with weak coalitional monotonicity in Theorem 2.4(ii). In this case, proportional-balanced treatment even can be weakened by requiring the proportionality only for games in which all players are weak symmetric.

- **Weak proportional-balanced treatment.** For all $(N, v) \in \mathcal{G}_{nz}^N$ such that all players are weak symmetric in (N, v), it holds that $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for all $i, j \in N$.

Theorem 2.5. *The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, weak proportional-balanced treatment, and weak coalitional monotonicity.*

Notice that by using the monotonicity axioms in Theorems 2.4 and 2.5, we can get rid of weak linearity.

Considering the relationship between our monotonicity axioms and the stronger versions introduced in Casajus and Huettner (2014a) and van den Brink (2007) (to characterize the ESD value or the ED value, see Theorems 1.6, 1.7, and 1.8), from Theorems 2.4 and 2.5, we obtain the following corollary.

Corollary 2.1. Let $|N| \geq 2$. There is no value on $\mathcal{G}_{nz}^N$ satisfying

(i) efficiency, proportional-balanced treatment, and coalitional surplus equivalence.

(ii) efficiency, proportional-balanced treatment, and coalitional surplus monotonicity.

(iii) efficiency, weak proportional-balanced treatment, and coalitional monotonicity.

As shown before, weak coalitional surplus monotonicity is stronger than both weak coalitional surplus equivalence and weak coalitional monotonicity. We conclude this subsection by mentioning two values to show logical independence of weak coalitional surplus equivalence and weak coalitional monotonicity. The value $\psi_i(N,v) = v(\{i\}) - \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]$, $i \in N$, satisfies weak coalitional surplus equivalence, but not weak coalitional monotonicity; the ED value $\psi_i(N,v) = \frac{v(N)}{n}$, $i \in N$, satisfies weak coalitional monotonicity, but not weak coalitional surplus equivalence.

2.3.3 Consistency

In this subsection, we consider a variable player set, and characterize the PD value by *proportional standardness* used in Ortmann (2000), Khmelnitskaya and Driessen (2003), and Huettner (2015), and *projection consistency* used in Funaki and Yamato (2001), van den Brink and Funaki (2009), van den Brink et al. (2016), Calleja and Llerena (2017), and Calleja and Llerena (2019).

The consistency principle is based on the idea that the payoffs of the remaining players should not change when a specific player leaves the game with the payoff that is assigned to this player by the solution. In the literature, there appear various reduced games which assess the effect on the worths of coalitions of remaining players in a different way. Here, we consider the projection reduced game, which is also mentioned in Subsection 1.3.5.

If a player $j \in N$ leaves game (N,v) with a certain payoff, then the projection reduced game is a game on the remaining player set that assigns to every proper subset of $N\backslash\{j\}$ its worth in the original game, and to coalition $N\backslash\{j\}$ assigns its worth in (N,v) minus the payoff assigned to player j.

Definition 2.2. Given a game $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 2$, a player $j \in N$ and a payoff vector $x \in \mathbb{R}^N$, the *projection reduced game with respect to j and x* is the game $(N\backslash\{j\}, v^x)$ given by

$$v^x(S) = \begin{cases} v(N) - x_j & \text{if } S = N\backslash\{j\}, \\ v(S) & \text{if } S \subset N\backslash\{j\}. \end{cases}$$

Projection consistency requires that the payoffs assigned to the remaining players in $N\backslash\{j\}$, after player j leaving the game with its payoff according to a value ψ, is the same in the reduced game as in the original game.

Definition 2.3. A value ψ satisfies *projection consistency* if for every game $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N,v)$, it holds that $(N\backslash\{j\}, v^x) \in \mathcal{G}_{nz}$, and $\psi_i(N\backslash\{j\}, v^x) = \psi_i(N,v)$ for all $i \in N\backslash\{j\}$.

Definition 2.4. A value ψ satisfies *proportional standardness* if for every game $(N,v) \in \mathcal{G}_{nz}$ with $|N| = 2$, it holds that

$$\psi_i(N,v) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})}[v(N) - v(\{i\}) - v(\{j\})], \ \ \forall i,j \in N.$$

This is equivalent to $\psi_i(N,v) = \frac{v(\{i\})}{v(\{i\})+v(\{j\})}v(N)$. Proportional standardness is called "proportional for two person games" in Ortmann (2000).

Proposition 2.2. *The PD value on $\mathcal{G}_{nz}$ satisfies projection consistency.*

Projection consistency together with proportional standardness (for two-player games) characterizes the PD value on the class of games with at least two players. We denote the class of games in $\mathcal{G}_{nz}$ with at least two players by $\mathcal{G}_{nz}^{\geq 2}$.

Theorem 2.6. *The PD value is the unique value on $\mathcal{G}_{nz}^{\geq 2}$ that satisfies proportional standardness and projection consistency.*

Replacing proportional standardness by standardness in Theorem 2.6 yields a characterization of the equal surplus division value, as a special case of Theorem 4.4 in van den Brink et al. (2016).

Proposition 2.1 and Theorem 2.6 together imply the following corollary.

Corollary 2.2. The PD value is the unique value on $\mathcal{G}_{nz}^{\geq 2}$ that satisfies efficiency, proportional-balanced treatment, and projection consistency.

Due to efficiency, this corollary also holds on $\mathcal{G}_{nz}$.

2.3.4 Characterizations for two-player games

In Subsection 2.3.3 we imposed proportional standardness to characterize the PD value for any player set. Note that proportional standardness, as other two-player standardness axiom, is a quite strong axiom since it coincides with the definition of the PD value for two-player game. In this subsection, we support proportional standardness by showing how the PD value can be characterized on the class of two-player games. We first characterize the PD value for rational numbers, and then apply continuity to obtain a characterization for real worths. Denote $\mathcal{G}_{nz}^2 = \{(N,v) \in \mathcal{G}_{nz} \mid |N| = 2\}$ and $\mathcal{G}_{nz\mathbb{Q}}^2 = \{(N,v) \in \mathcal{G}_{nz}^2 \mid v(S) \in \mathbb{Q} \text{ for all } S \subseteq N\}$, so the worths of coalitions in games in $\mathcal{G}_{nz\mathbb{Q}}^2$ are rational numbers. In addition, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the sets of integers, positive integers and negative integers, respectively.

We begin this subsection by introducing two additional axioms, the first on $\mathcal{G}_{nz\mathbb{Q}}^2$ and the second on $\mathcal{G}_{nz}^2$.

- **Grand worth additivity**. For games $(N,v),(N,w) \in \mathcal{G}_{nz\mathbb{Q}}^2$ with $N = \{i,j\}$ such that $v(\{i\}) = w(\{i\})$ and $v(\{j\}) = w(\{j\})$, it holds that $\psi(N,v) + \psi(N,w) = \psi(N, v \oplus w)$, where $(N, v \oplus w)$ is defined as: $(v \oplus w)(\{i\}) = v(\{i\})$, $(v \oplus w)(\{j\}) = v(\{j\})$ and $(v \oplus w)(N) = v(N) + w(N)$.

- **Inessential game property for two-player games**. For every game $(N, v) \in \mathcal{G}^2_{nz}$ with $N = \{i, j\}$ such that $v(\{i\}) + v(\{j\}) = v(\{i, j\})$, it holds that $\psi_i(N, v) = v(\{i\})$ and $\psi_j(N, v) = v(\{j\})$.

Grand worth additivity states that for two games in which all worths are rational numbers and the stand-alone worths are the same, we consider the game where the stand-alone worths are the same as in the original game, and the worth of the grand coalition equals the sum of the worths of the grand coalition in the two games, then the payoff to each player equals the sum of the payoffs in the two separate games. This axiom is similar to additivity in Moulin (1987) and Chun (1988) for bankruptcy problems. The inessential game property is a well-known axiom requiring that players earn their stand-alone payoff in an inessential game. In this subsection we require the inessential game property only for two-player games.

First, we show that these two axioms characterize the PD value on the class of two-player games with rational worths.

Proposition 2.3. *The PD value is the unique value on $\mathcal{G}^2_{nz\mathbb{Q}}$ that satisfies grand worth additivity and the inessential game property for two-player games.*

Next, adding *continuity for two-player games*, which states that if two games are almost the same then their payoffs are almost the same, we can extend this result from rational numbers to real numbers.

- **Continuity for two-player games**. For all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{G}^2_{nz}$ such that $\lim_{k\to\infty} (N, w_k) = (N, v)$, it holds that $\lim_{k\to\infty} \psi(N, w_k) = \psi(N, v)$.

Theorem 2.7. *The PD value is the unique value on $\mathcal{G}^2_{nz}$ that satisfies grand worth additivity, the inessential game property for two-player games, and continuity for two-player games.*

By Theorems 2.6 and 2.7, we immediately obtain the following corollary.

Corollary 2.3. The PD value is the unique value on $\mathcal{G}^{\geq 2}_{nz}$ that satisfies grand worth additivity, the inessential game property, continuity for two-player games, and projection consistency.

Corollary 2.3 is valid on $\mathcal{G}_{nz}$ if we require the inessential game property for all games in $\mathcal{G}_{nz}$, i.e., for every game $(N, v) \in \mathcal{G}_{nz}$ such that $v(S) = \sum_{i \in S} v(\{i\})$, it holds that $\psi_i(N, v) = v(\{i\})$ for all $i \in N$.

Remark 2.5. One can easily check that each of the following two axioms together with the inessential player property on $\mathcal{G}^2_{nz}$ characterize the PD value.

- **Grand worth proportionality**. For two games $(N, v), (N, w) \in \mathcal{G}^2_{nz}$ and $\alpha \in \mathbb{R}$ such that $N = \{i, j\}$, $v(\{i\}) = w(\{i\})$, $v(\{j\}) = w(\{j\})$ and $w(N) = \alpha v(N)$, it holds that $\psi(N, w) = \alpha \psi(N, v)$.

- **Relational covariance**. For two games $(N,v),(N,w) \in \mathcal{G}^2_{nz}$ and $\alpha \in \mathbb{R}$ such that $N = \{i,j\}$, $v(\{i\}) = w(\{i\})$, $v(\{j\}) = w(\{j\})$ and $w(N) = v(N) + \alpha[v(\{i\}) + v(\{j\})]$, it holds that $\psi_i(N,w) = \psi_i(N,v) + \alpha v(\{i\})$ and $\psi_j(N,w) = \psi_j(N,v) + \alpha v(\{j\})$.

From Remark 2.5 and Theorem 2.6, we have that the PD value on $\mathcal{G}^{\geq 2}_{nz}$ is characterized by the inessential game property, projection consistency, and either grand worth proportionality or relational covariance.

Remark 2.6. Ortmann (2000) introduced his proportional value [6] that can be characterized by proportional standardness and consistency due to Hart and Mas-Colell (1989). As a consequence, characterizations of Ortmann's proportional value can be obtained by replacing proportional standardness by the axioms in Theorem 2.7 or Remark 2.5. Notice that we cannot use the axiomatization as given by Proposition 2.1, since proportional-balanced treatment is not satisfied by Ortmann's proportional value for games with more than two players.

2.4 Independence of axioms

Logical independence of the axioms used in the characterization results can be shown by the following alternative values.

Theorem 2.3:

(i) The value ψ on $\mathcal{G}^N_{nz}$ defined for all $(N,v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, by

$$\psi_i(N,v) = v(\{i\}) \tag{2.7}$$

satisfies all axioms, but not efficiency.

(ii) Let D_v be the set of all dummy players and all dummifying players in (N,v). The value ψ on $\mathcal{G}^N_{nz}$ defined for each $(N,v) \in \mathcal{G}^N_{nz}$ and each $i \in N$, by

$$\psi_i(N,v) = \begin{cases} v(\{i\}), & \text{if } i \in D_v, \\ \frac{v(\{i\})}{\sum_{j\in N\setminus D_v} v(\{j\})}\left[v(N) - \sum_{j\in D_v} v(\{j\})\right], & \text{otherwise.} \end{cases}$$

satisfies all axioms, but not weak linearity.

(iii) The ESD value on $\mathcal{G}^N_{nz}$ satisfies all axioms, but not proportional-balanced treatment.

[6] Ortmann's proportional value is recursively defined for all $(N,v) \in \mathcal{G}_{nz+}$ and $i \in S$ by

$$\psi_i(S,v|_S) = v(S)\left(1 + \sum_{j\in S\setminus\{i\}} \frac{\psi_j(S\setminus\{i\},v|_{S\setminus\{i\}})}{\psi_i(S\setminus\{j\},v|_{S\setminus\{j\}})}\right)^{-1}$$

if $|S| > 1$, and $\psi_i(S,v|_S) = v(\{i\})$ if $|S| = 1$. Here, for any $S \subseteq N$, $(S,v|_S)$ is defined by $v|_S(T) = v(T)$ for all $T \subseteq S$.

(iv) The value defined for all $(N, v) \in \mathcal{G}_{nz}^N$ and all $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - v(N\backslash\{i\}) + \sum_{h\in N} \frac{v(\{h\})v(N\backslash\{h\})}{\sum_{k\in N} v(\{k\})}\right] \tag{2.8}$$

satisfies all axioms, but not the dummifying player property. Clearly, (2.8) coincides with (2.2) by taking $\lambda_S = \frac{\sum_{j\in S} v(\{j\}) \sum_{j\in N\backslash S} v(\{j\})}{(\sum_{k\in N} v(\{k\}))^2}$ for all $S \subset N$ with $|S| = n-1$, and $\lambda_S = 0$ otherwise.

Remark 2.7. Notice that (2.8) also coincides with (2.6) by taking $\psi'(N,v) = EANSC(N,v)$, see (1.2). The calculation is shown as follows. Substituting the EANSC value into (2.6) , we have

$$\begin{aligned}
\psi_i(N,v) =& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\psi_i'(N,v) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - \sum_{h\in N}\frac{v(\{h\})\psi_h'(N,v)}{\sum_{k\in N} v(\{k\})}\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[SC_i + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right] + \\
& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - \sum_{h\in N}\frac{v(\{h\})\left[SC_h + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right]}{\sum_{k\in N} v(\{k\})}\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[SC_i + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right] + \\
& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - \sum_{h\in N}\frac{v(\{h\})SC_h}{\sum_{k\in N} v(\{k\})} - \sum_{h\in N}\frac{v(\{h\})}{\sum_{k\in N} v(\{k\})}\frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[SC_i + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right] + \\
& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - \sum_{h\in N}\frac{v(\{h\})SC_h}{\sum_{k\in N} v(\{k\})} - \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[SC_i + v(N) - \sum_{h\in N}\frac{v(\{h\})SC_h}{\sum_{k\in N} v(\{k\})}\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - v(N\backslash\{i\}) + v(N) - \sum_{h\in N}\frac{v(\{h\})[v(N) - v(N\backslash\{h\})]}{\sum_{k\in N} v(\{k\})}\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - v(N\backslash\{i\}) + v(N) - v(N) + \sum_{h\in N}\frac{v(\{h\})v(N\backslash\{h\})}{\sum_{k\in N} v(\{k\})}\right] \\
=& \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - v(N\backslash\{i\}) + \sum_{h\in N}\frac{v(\{h\})v(N\backslash\{h\})}{\sum_{k\in N} v(\{k\})}\right],
\end{aligned}$$

which coincides with (2.8).

Theorem 2.4:

(i) The value defined by (2.7) satisfies all axioms, but not efficiency.

(ii) The ESD value on $\mathcal{G}_{nz}^N$ satisfies all axioms, but not proportional-balanced treatment.

(iii) The value defined by (2.8) satisfies all axioms, but neither weak coalitional surplus equivalence nor weak coalitional surplus monotonicity.

Theorem 2.5:

(i) The value defined by (2.7) satisfies all axioms, but not efficiency.

(ii) The ED value on $\mathcal{G}_{nz}^N$ satisfies all axioms, but not weak proportional-balanced treatment.

(iii) The value defined by (2.8) satisfies all axioms, but not weak coalitional monotonicity.

Theorem 2.6:

(i) Ortmann's proportional value satisfies proportional standardness, but not projection consistency.

(ii) The ESD value on $\mathcal{G}_{nz}^N$ satisfies projection consistency, but not proportional standardness.

Theorem 2.7:

(i) The value defined by (2.7) satisfies all axioms, but not grand worth additivity.

(ii) The ED value on $\mathcal{G}_{nz}^2$ satisfies all axioms, but not the inessential game property for two-player games.

(iii) The value ψ defined for all $(N,v) \in \mathcal{G}_{nz}^N$ and all $i \in N$, by

$$\psi_i(N,v) = \begin{cases} PD_i(N,v), & \text{if } v(N) \in \mathbb{Q}, \\ ESD_i(N,v), & \text{if } v(N) \notin \mathbb{Q}. \end{cases}$$

satisfies all axioms, but not continuity for two-player games.

2.5 Proofs

Proof of Proposition 2.1. It is clear that the PD value satisfies the two axioms. Conversely, let ψ be a value satisfying the two axioms. Since the two players in a two-player game $(N,v) \in \mathcal{G}_{nz}^N$ are always weak symmetric, by proportional-balanced treatment, $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for $i,j \in N$. By efficiency, $\psi_i(N,v) + \psi_j(N,v) = v(N)$. Therefore, we obtain $\psi_k(N,v) = \frac{v(\{k\})}{v(\{i\})+v(\{j\})}v(N)$, $k \in \{i,j\}$, as desired. □

Proof of Theorem 2.1. *Existence*: It is straightforward to show that any value defined by (2.2) satisfies efficiency and weak linearity. Next, we show that it also satisfies proportional-balanced treatment. Let $i, k \in N$ be two players such that $v(S \cup \{i\}) = v(S \cup \{k\})$ for all $S \subseteq N \backslash \{i,k\}$, $S \neq \emptyset$. We have

$$
\begin{aligned}
&\frac{\psi_i(N,v)}{v(\{i\})} - \frac{\psi_k(N,v)}{v(\{k\})} \\
= &\sum_{\substack{S:i\in S, k\notin S\\|S|\geq 2}} \frac{\lambda_S}{\sum_{j\in S} v(\{j\})} v(S) - \sum_{\substack{S:i\notin S, k\in S\\|S|\geq 2}} \frac{\lambda_S}{\sum_{j\in N\backslash S} v(\{j\})} v(S) \\
&- \Big[\sum_{\substack{S:k\in S, i\notin S\\|S|\geq 2}} \frac{\lambda_S}{\sum_{j\in S} v(\{j\})} v(S) - \sum_{\substack{S:k\notin S, i\in S\\|S|\geq 2}} \frac{\lambda_S}{\sum_{j\in N\backslash S} v(\{j\})} v(S)\Big] \\
= &\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{i\}}}{\sum_{j\in S} v(\{j\})+v(\{i\})} v(S\cup\{i\}) - \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{k\}}}{\sum_{j\in N\backslash S} v(\{j\})-v(\{k\})} v(S\cup\{k\}) \\
&- \Big[\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{k\}}}{\sum_{j\in S} v(\{j\})+v(\{k\})} v(S\cup\{k\}) - \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{i\})}}{\sum_{j\in N\backslash S} v(\{j\})-v(\{i\})} v(S\cup\{i\}\Big] \\
= &\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{i\}}}{\sum_{j\in S} v(\{j\})+v(\{i\})} v(S\cup\{i\}) + \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{i\}}}{\sum_{j\in N\backslash S} v(\{j\})-v(\{i\})} v(S\cup\{i\}) \\
&- \Big[\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{k\}}}{\sum_{j\in N\backslash S} v(\{j\})-v(\{k\})} v(S\cup\{k\}) + \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\lambda_{S\cup\{k\}}}{\sum_{j\in S} v(\{j\})+v(\{k\})} v(S\cup\{k\})\Big] \\
= &\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\sum_{j\in N} v(\{j\})\lambda_{S\cup\{i\}} v(S\cup\{i\})}{[\sum_{j\in S} v(\{j\})+v(\{i\})][\sum_{j\in N\backslash S} v(\{j\})-v(\{i\})]} - \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\sum_{j\in N} v(\{j\})\lambda_{S\cup\{k\}} v(S\cup\{k\})}{[\sum_{j\in S} v(\{j\})+v(\{k\})][\sum_{j\in N\backslash S} v(\{j\})-v(\{k\})]} \\
= &\sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\sum_{j\in N} v(\{j\})\lambda_{S\cup\{k\}} v(S\cup\{i\})}{[\sum_{j\in S} v(\{j\})+v(\{k\})][\sum_{j\in N\backslash S} v(\{j\})-v(\{k\})]} - \sum_{\substack{S:i,k\notin S\\|S|\geq 1}} \frac{\sum_{j\in N} v(\{j\})\lambda_{S\cup\{k\}} v(S\cup\{k\})}{[\sum_{j\in S} v(\{j\})+v(\{k\})][\sum_{j\in N\backslash S} v(\{j\})-v(\{k\})]} \\
= &0,
\end{aligned}
$$

where the fifth equality follows by (2.3). Thus, (2.2) satisfies proportional-balanced treatment.

Uniqueness: Let ψ be a value satisfying efficiency, weak linearity, and proportional-balanced treatment. For $|N| = 1$ and $|N| = 2$, uniqueness follows from efficiency and Proposition 2.1, respectively. Now let $(N, v) \in \mathcal{G}^N_{nz}$ be an arbitrary game with $|N| \geq 3$. In order to use the property of weak linearity, we decompose (N, v) into the unique combination of the following two kinds of games (N, w) and (N, w^S) [7]. The game (N, w) is defined as follows:

$$
w(T) = \begin{cases} v(\{i\}), & \text{if } T = \{i\} \text{ for all } i \in N, \\ 0, & \text{otherwise.} \end{cases}
$$

[7]For any $(N, v) \in \mathcal{G}^N_{nz}$, the collection of games $\{(N, w), (N, w^S)_{S\subseteq N, |S|\geq 2}\}$ is a basis of the class of games $\mathcal{G}^N_v = \{(N, v') \in \mathcal{G}^N_{nz} \mid \exists c \in \mathbb{R} \text{ such that } v'(\{i\}) = cv(\{i\}) \text{ for all } i \in N\} \cup \{(N, v) \in \mathcal{G}^N \mid v(\{i\}) = 0 \text{ for all } i \in N\}$. The dimension of $\mathcal{G}^N_v$ is $2^n - n$. Another interesting basis can be found in the proof of Proposition 5 in Béal et al. (2018) or in van den Brink et al. (2020).

For any coalition $S \subseteq N$ with $|S| \geq 2$, the game (N, w^S) is defined as follows:

$$w^S(T) = \begin{cases} v(\{i\}), & \text{if } T = \{i\} \text{ for all } i \in N, \\ 1, & \text{if } T = S, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that (N, v) can be written as $v = I(v)w + \sum_{S \subseteq N, |S| \geq 2} v(S)w^S$, where $I(v) = 1 - \sum_{S \subseteq N, |S| \geq 2} v(S)$. By weak linearity[8], we have

$$\psi_i(N, v) = I(v)\psi_i(N, w) + \sum_{S \subseteq N, |S| \geq 2} v(S)\psi_i(N, w^S) \quad \text{for all } i \in N.$$

Now, by proportional-balanced treatment, for each $S \subset N$ with $|S| \geq 2$, since all players in S are weak symmetric in (N, w^S), and the same for all players in $N \backslash S$, there must exist some λ_S and μ_S such that

$$\psi_i(N, w^S) = \begin{cases} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \lambda_S, & \text{if } i \in S, \\ \frac{v(\{i\})}{\sum_{j \in N \backslash S} v(\{j\})} \mu_S, & \text{if } i \notin S. \end{cases} \tag{2.9}$$

By efficiency, it must be $\sum_{i \in S} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \lambda_S + \sum_{i \in N \backslash S} \frac{v(\{i\})}{\sum_{j \in N \backslash S} v(\{j\})} \mu_S = 0$, which shows $\lambda_S = -\mu_S$. Similarly, $\psi_i(N, w^N) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}$ for all $i \in N$. Meanwhile, we have $\psi(N, w) = 0$. Putting all together we have the expression of ψ as given by (2.2).

Accordingly, let us see that λ_S only depends on the size of S ($S \neq N$) and the worths of all singleton coalitions $\{i\}$, $i \in N$. Let $S \subset N$, $S \neq \emptyset$ with $i, j \notin S$, and consider the game $(N, w^{S \cup \{i\}} + w^{S \cup \{j\}})$. In this game, since i and j are weak symmetric, it must be that $\frac{1}{v(\{i\})} \psi_i(N, w^{S \cup \{i\}} + w^{S \cup \{j\}}) = \frac{1}{v(\{j\})} \psi_j(N, w^{S \cup \{i\}} + w^{S \cup \{j\}})$. With (2.9) and weak linearity, we have that

$$\begin{aligned} &\frac{1}{\sum_{k \in S \cup \{i\}} v(\{k\})} \lambda_{S \cup \{i\}} - \frac{1}{\sum_{k \in N \backslash (S \cup \{j\})} v(\{k\})} \lambda_{S \cup \{j\}} \\ =&\frac{1}{\sum_{k \in S \cup \{j\}} v(\{k\})} \lambda_{S \cup \{j\}} - \frac{1}{\sum_{k \in N \backslash (S \cup \{i\})} v(\{k\})} \lambda_{S \cup \{i\}}, \end{aligned}$$

from which it immediately follows that

$$\frac{\lambda_{S \cup \{i\}}}{\sum_{k \in S \cup \{i\}} v(\{k\}) \sum_{k \in N \backslash (S \cup \{i\})} v(\{k\})} = \frac{\lambda_{S \cup \{j\}}}{\sum_{k \in S \cup \{j\}} v(\{k\}) \sum_{k \in N \backslash (S \cup \{j\})} v(\{k\})}.$$

Therefore, whenever S and T are of the same size, replacing player by player, we can form a sequence with at most $s + 1$ coalitions, such that the first one is S, and any of them is the result of replacing a player of S by a player of $N \backslash S$. In this way, we

[8]To ensure that we stay in the class $\mathcal{G}^N_{nz}$, we should consider the games in which their coefficients are nonzero in a suitable ordering, just like the technical approach as given by Lemma 5 in Béal et al. (2018).

conclude the relationship between λ_S and λ_T given by (2.3). $\square$

Proof of Theorem 2.2. Let ψ be a value satisfying these axioms. First, suppose that $|N| \geq 4$. Consider any game $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$ being a dummy player in (N,v). By Theorem 2.1, we have

$$\begin{aligned}
\psi_i(N,v) =& \tfrac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) + \sum_{\substack{S:i\in S\neq N\\|S|\geq 2}} \tfrac{v(\{i\})\cdot\lambda_S}{\sum_{j\in S} v(\{j\})} v(S) - \sum_{\substack{S:i\notin S\\|S|\geq 2}} \tfrac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)\\
=& \tfrac{v(\{i\})v(N)}{\sum_{j\in N} v(\{j\})} + \sum_{\substack{S:i\notin S\\1\leq|S|\leq n-2}} \tfrac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(S\cup\{i\}) - \sum_{\substack{S:i\notin S\\|S|\geq 2}} \tfrac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)\\
=& \tfrac{v(\{i\})[v(N\setminus\{i\})+v(\{i\})]}{\sum_{j\in N} v(\{j\})} + \sum_{j\in N\setminus\{i\}} \tfrac{v(\{i\})\cdot\lambda_{\{i,j\}}}{v(\{i\})+v(\{j\})} v(\{i,j\}) - \lambda_{N\setminus\{i\}} v(N\setminus\{i\})\\
&+ \sum_{\substack{S:i\notin S\\2\leq|S|\leq n-2}} [\tfrac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(S\cup\{i\}) - \tfrac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)]\\
=& \tfrac{v(\{i\})v(\{i\})}{\sum_{j\in N} v(\{j\})} + v(\{i\}) \sum_{j\in N\setminus\{i\}} \lambda_{\{i,j\}} + [\tfrac{v(\{i\})}{\sum_{j\in N} v(\{j\})} - \lambda_{N\setminus\{i\}}] v(N\setminus\{i\})\\
&+ \sum_{\substack{S:i\notin S\\2\leq|S|\leq n-2}} \tfrac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(\{i\}) + \sum_{\substack{S:i\notin S\\2\leq|S|\leq n-2}} [\tfrac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} - \tfrac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})}] v(S),
\end{aligned}$$

where the third equality follows from i being a dummy player in (N,v).

Since, by the dummy player property, the payoff of dummy player i should not depend on $v(S)$, $i \notin S$ and $2 \leq |S| \leq n-1$, the third term and the fifth term of the above equation must be equal to 0, which yields

$$\lambda_{N\setminus\{i\}} = \tfrac{v(\{i\})}{\sum_{j\in N} v(\{j\})}, \tag{2.10}$$

$$\frac{\lambda_{S\cup\{i\}}}{\lambda_S} = \frac{\sum_{j\in S\cup\{i\}} v(\{j\})}{\sum_{j\in N\setminus S} v(\{j\})} \quad \text{for } S \subset N \text{ with } 2 \leq |S| \leq n-2. \tag{2.11}$$

Since λ_S for each $S \subset N$ with $|S| \geq 2$ satisfies (2.3), then (2.10) and (2.3) together imply that, for any $k_1 \in N\setminus\{i\}$,

$$\lambda_{N\setminus\{k_1\}} = \frac{v(\{k_1\}) \sum_{j\in N\setminus\{k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{i\}} v(\{j\}) \sum_{j\in N} v(\{j\})}.$$

By using (2.11), we have

$$\begin{aligned}
\lambda_{N\setminus\{i,k_1\}} &= \frac{\sum_{j\in\{i,k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{k_1\}} v(\{j\})} \lambda_{N\setminus\{k_1\}}\\
&= \frac{v(\{k_1\}) \sum_{j\in\{i,k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{i\}} v(\{j\}) \sum_{j\in N} v(\{j\})}.
\end{aligned}$$

The above equation together with (2.3) imply that, for any $k_2 \in N\backslash\{i,k_1\}$,

$$\begin{aligned}\lambda_{N\backslash\{k_1,k_2\}} &= \frac{\sum_{j\in N\backslash\{k_1,k_2\}} v(\{j\}) \sum_{j\in\{k_1,k_2\}} v(\{j\})}{\sum_{j\in N\backslash\{i,k_1\}} v(\{j\}) \sum_{j\in\{i,k_1\}} v(\{j\})}\lambda_{N\backslash\{i,k_1\}} \\ &= \frac{\sum_{j\in N\backslash\{k_1,k_2\}} v(\{j\}) \sum_{j\in\{k_1,k_2\}} v(\{j\}) v(\{k_1\})}{\sum_{j\in N\backslash\{i,k_1\}} v(\{j\}) \sum_{j\in N\backslash\{i\}} v(\{j\}) \sum_{j\in N} v(\{j\})}.\end{aligned}$$

Now, exchanging the order of k_1 and k_2, we have

$$\lambda_{N\backslash\{k_1,k_2\}} = \frac{\sum_{j\in N\backslash\{k_1,k_2\}} v(\{j\}) \sum_{j\in\{k_1,k_2\}} v(\{j\}) v(\{k_2\})}{\sum_{j\in N\backslash\{i,k_2\}} v(\{j\}) \sum_{j\in N\backslash\{i\}} v(\{j\}) \sum_{j\in N} v(\{j\})}.$$

Therefore, it must be that

$$\frac{v(\{k_1\})}{\sum_{j\in N\backslash\{i,k_1\}} v(\{j\})} = \frac{v(\{k_2\})}{\sum_{j\in N\backslash\{i,k_2\}} v(\{j\})},$$

from which it follows that $v(\{k_1\}) = v(\{k_2\})$ for any $k_1, k_2 \in N\backslash\{i\}$, i.e. all stand-alone worths except $v(\{i\})$ must be the same. This contradicts the definition of $\mathcal{G}^N_{nz}$.

Next, suppose that $|N| = 3$. Consider $(N,v) \in \mathcal{G}^N_{nz}$ with $N = \{i,j,k\}$ and $i \in N$ such that i is a dummy player in (N,v). By Theorem 2.1 and i being a dummy player, we have

$$\begin{aligned}\psi_i(N,v) =& \frac{v(\{i\})[v(\{j,k\})+v(\{i\})]}{v(\{i\})+v(\{j\})+v(\{k\})} + v(\{i\})\left(\frac{\lambda_{\{i,j\}}(v(\{i\})+v(\{j\}))}{v(\{i\})+v(\{j\})} + \frac{\lambda_{\{i,k\}}(v(\{i\})+v(\{k\}))}{v(\{i\})+v(\{k\})} - \frac{\lambda_{\{j,k\}}v(\{j,k\})}{v(\{i\})}\right) \\ =& \frac{v(\{i\})[v(\{j,k\})+v(\{i\})]}{v(\{i\})+v(\{j\})+v(\{k\})} + v(\{i\})(\lambda_{\{i,j\}} + \lambda_{\{i,k\}}) - \lambda_{\{j,k\}}v(\{j,k\}) \\ =& v(\{j,k\})\left(\frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} - \lambda_{\{j,k\}}\right) \\ &+ v(\{i\})\left(\frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} + \lambda_{\{i,j\}} + \lambda_{\{i,k\}}\right).\end{aligned}$$

By the dummy player property, we have $\psi_i(N,v) = v(\{i\})$. Since $\psi_i(N,v)$ should not depend on $v(\{j,k\})$, it must be that $\frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} - \lambda_{\{j,k\}} = 0$, and thus

$$\lambda_{\{j,k\}} = \frac{v(\{i\})}{v(\{i\}) + v(\{j\}) + v(\{k\})}. \tag{2.12}$$

But then

$$\frac{v(\{i\})}{v(\{i\}) + v(\{j\}) + v(\{k\})} + \lambda_{\{i,j\}} + \lambda_{\{i,k\}} = 1,$$

implying that

$$\lambda_{\{i,j\}} + \lambda_{\{i,k\}} = \frac{v(\{j\}) + v(\{k\})}{v(\{i\}) + v(\{j\}) + v(\{k\})}. \tag{2.13}$$

Meanwhile, (2.3) implies that

$$\frac{\lambda_{\{j,k\}}}{[v(\{j\})+v(\{k\})]v(\{i\})}=\frac{\lambda_{\{i,j\}}}{[v(\{i\})+v(\{j\})]v(\{k\})}=\frac{\lambda_{\{i,k\}}}{[v(\{i\})+v(\{k\})]v(\{j\})}.$$

It follows that

$$\frac{\lambda_{\{j,k\}}}{[v(\{j\})+v(\{k\})]v(\{i\})}=\frac{\lambda_{\{i,j\}}+\lambda_{\{i,k\}}}{[v(\{i\})+v(\{j\})]v(\{k\})+[v(\{i\})+v(\{k\})]v(\{j\})}. \tag{2.14}$$

Substituting (2.12) and (2.13) into (2.14) yields

$$(v(\{j\}))^2+(v(\{k\}))^2=v(\{i\})v(\{k\})+v(\{i\})v(\{j\}),$$

which does not hold for all games in $\mathcal{G}^N_{nz}$. □

Proof of Theorem 2.3. It is obvious that the PD value satisfies efficiency, weak linearity, proportional-balanced treatment, and the dummifying player property. It remains to prove the uniqueness part. Let ψ be a value satisfying these axioms. By Theorem 2.1, any value satisfying efficiency, weak linearity and proportional-balanced treatment is given by (2.2) for some λ_S ($S \subseteq N, |S| \geq 2$) satisfying (2.3). To derive λ_S, we consider a modified game $(N, v_i) \in \mathcal{G}^N_{nz}$ with respect to $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$, defined by

$$v_i(S)=\begin{cases} v(\{j\}), & \text{if } S=\{j\} \text{ for all } j \in N,\\ \sum_{j\in S} v(\{j\}), & \text{if } i \in S \text{ and } |S|\geq 2,\\ v(S), & \text{otherwise.}\end{cases}$$

Applying (2.2) to the game (N, v_i), we have

$$\psi_i(N,v_i)=v(\{i\})+v(\{i\})\Bigg[\sum_{\substack{S:i\in S\neq N\\|S|\geq 2}}\lambda_S-\sum_{\substack{S:i\notin S\\|S|\geq 2}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\Bigg].$$

Since i is dummifying in (N, v_i), the dummifying player property requires that $\psi_i(N, v_i) = v(\{i\})$, and thus

$$\sum_{\substack{S:i\in S\neq N\\|S|\geq 2}}\lambda_S-\sum_{\substack{S:i\notin S\\|S|\geq 2}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\equiv 0.$$

It follows that

$$\sum_{s=2}^{n-1}\Big[\sum_{\substack{S:i\in S\\|S|=s}}\lambda_S-\sum_{\substack{S:i\notin S\\|S|=s}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\Big]\equiv 0. \tag{2.15}$$

We will show that $\lambda_S = 0$ for all $S \subset N$ in (2.15). Suppose by contradiction that there exist some $S \subset N$ with $s \in \{2, \ldots, n-1\}$ such that $\lambda_S \neq 0$ and $|S| = s$. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$ be the set of such coalitional sizes. Note that (2.3) implies that if $\lambda_S \neq 0$, then all coefficients of coalitions of the same size s are not equal to zero. We denote by $\mathcal{S}_k = \{S_k^1, S_k^2, \ldots, S_k^h\}$, $k = 1, \ldots, m$, $h = \binom{n}{s_k}$, the set of all coalitions of the same size $s_k \in \mathcal{S}$. Pick any $S_k^r \in \mathcal{S}_k$ with $i \in S_k^r$. By (2.3), we have $\lambda_{S_k^t} = \frac{\sum_{j \in S_k^t} v(\{j\}) \sum_{j \in N \setminus S_k^t} v(\{j\})}{\sum_{j \in S_k^r} v(\{j\}) \sum_{j \in N \setminus S_k^r} v(\{j\})} \lambda_{S_k^r}$ for any $S_k^t \in \mathcal{S}_k$ (it obviously holds for the case $S_k^t = S_k^r$). With this equality, (2.15) can be written as

$$\sum_{s_k \in \mathcal{S}} \Big[A(\mathcal{S}_k) - \sum_{S_k^t \in \mathcal{S}_k, i \notin S_k^t} B(S_k^t) v(S_k^t)\Big] \lambda_{S_k^r} = 0, \tag{2.16}$$

where $A(\mathcal{S}_k) = \sum_{S_k^t \in \mathcal{S}_k, i \in S_k^t} \frac{\sum_{j \in S_k^t} v(\{j\}) \sum_{j \in N \setminus S_k^t} v(\{j\})}{\sum_{j \in S_k^r} v(\{j\}) \sum_{j \in N \setminus S_k^r} v(\{j\})}$ and $B(S_k^t) = \frac{\sum_{j \in S_k^t} v(\{j\})}{\sum_{j \in S_k^r} v(\{j\}) \sum_{j \in N \setminus S_k^r} v(\{j\})}$.

Now, pick any $s_l \in \{s_1, s_2, \ldots, s_m\}$ and any $c \in \mathbb{R} \setminus \{0\}$, and consider the game $(N, v_{i,s_l}) \in \mathcal{G}_{nz}^N$ given by

$$v_{i,s_l}(S) = \begin{cases} v_i(S) + c, & \text{if } |S| = s_l \text{ and } i \notin S, \\ v_i(S), & \text{otherwise.} \end{cases}$$

Note that (2.3) shows that λ_S only depends on the size of S and the worths of all singleton coalitions. Therefore, since i is a dummifying player in (N, v_{i,s_l}), for this game we can obtain an equation similar as (2.16) but with an additional term that depends on c,

$$\sum_{s_k \in \mathcal{S}} \Big[A(\mathcal{S}_k) - \sum_{S_k^t \in \mathcal{S}_k, i \notin S_k^t} B(S_k^t) v(S_k^t)\Big] \lambda_{S_k^r} - c\lambda_{S_l^r} \sum_{S_l^t \in \mathcal{S}_l, i \notin S_l^t} B(S_l^t) = 0.$$

Together with this equation and (2.16), it holds that $-c\lambda_{S_l^r} \sum_{S_l^t \in \mathcal{S}_l, i \notin S_l^t} B(S_l^t) = 0$, yielding $\lambda_{S_l^r} = 0$, which is a contradiction. $\square$

Proof of Lemma 2.1. Suppose that value ψ satisfies weak coalitional surplus monotonicity. Consider a pair of games $(N, v), (N, v + w) \in \mathcal{G}_{nz}^N$, where $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ being a dummifying player such that $w(S) = \sum_{j \in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$. Since $(v + w)(S) - \sum_{j \in S}(v + w)(\{j\}) = v(S) - \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ with $i \in S$, by weak coalitional surplus monotonicity, we have $\psi_i(N, v + w) - (v + w)(\{i\}) = \psi_i(N, v) - v(\{i\})$. It follows that $\psi_i(N, v + w) = \psi_i(N, v) + w(\{i\})$, which shows that ψ satisfies weak coalitional surplus equivalence. $\square$

Proof of Theorem 2.4. (i) It is clear that the PD value satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus equivalence. Now, let ψ be a value on $\mathcal{G}_{nz}^N$ satisfying the three axioms. For $|N| = 1$, (2.1) is satisfied by efficiency. For

$|N| = 2$, (2.1) is obtained from Proposition 2.1. For $|N| \geq 3$, uniqueness follows by induction on $d(v) = |\{T \subseteq N \mid v(T) - \frac{1}{2}\sum_{j\in T} v(\{j\}) \neq 0 \text{ and } |T| \geq 2\}|$. For any $(N,v) \in \mathcal{G}^N_{nz}$, define $(N,v^0) \in \mathcal{G}^N_{nz}$ as follows:

$$v^0(T) = v(T) - \frac{1}{2}\sum_{j\in T} v(\{j\}) \quad \text{for all } T \subseteq N. \tag{2.17}$$

Intialization. If $d(v) = 0$, then $v(N) = \frac{1}{2}\sum_{j\in N} v(\{j\})$. Notice that, by $d(v) = 0$, in this case $v^0(T) = 0$ for all $T \subseteq N$ with $|T| \geq 2$. Clearly, all players $i,j \in N$ are weak symmetric in (N,v^0) and $v^0(N) = 0$. By efficiency and proportional-balanced treatment, we have $\psi_i(N,v^0) = 0$ for all $i \in N$. Notice that $(v - v^0)(\{i\}) = v(\{i\}) - v(\{i\}) + \frac{1}{2}v(\{i\}) = \frac{1}{2}v(\{i\})$ for all $i \in N$, and all players are dummifying in $(N, v - v^0)$ since $(v - v^0)(T) = v(T) - v(T) + \frac{1}{2}\sum_{j\in T} v(\{j\}) = \frac{1}{2}\sum_{j\in T} v(\{j\}) = \sum_{j\in T}(v - v^0)(\{j\})$. By weak coalitional surplus equivalence, $\psi_i(N,v) = \psi_i(N, v^0 + (v - v^0)) = \psi_i(N,v^0) + \frac{1}{2}v(\{i\})$ for all $i \in N$. Thus, $\psi_i(N,v) = \frac{1}{2}v(\{i\}) = PD_i(N,v)$ for all $i \in N$.

Proceeding by induction, assume that $\psi(N,w) = PD(N,w)$ for all $(N,w) \in \mathcal{G}^N_{nz}$ with $d(w) = h, 0 \leq h \leq 2^n - n - 2$. Consider $(N,v) \in \mathcal{G}^N_{nz}$ such that $d(v) = h+1$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{h+1}\}$ be the set of coalitions such that $v(S_k) - \frac{1}{2}\sum_{j\in S_k} v(\{j\}) \neq 0$ and $|S_k| \geq 2$. Let S be the intersection of all such coalitions S_k, i.e., $S = \bigcap\limits_{1\leq k\leq h+1} S_k$. We distinguish between two cases:

Case (a): $i \in N\backslash S$. Each player $i \in N\backslash S$ is a member of at most h coalitions in $\mathcal{S}$, and at least one $S_k \in \mathcal{S}$ such that $i \notin S_k$ (obviously, $S_k \neq N$). For $(N,v) \in \mathcal{G}^N_{nz}$, define three associated games as follows:

$$v^{S_k,1}(T) = \begin{cases} 0, & \text{if } T = S_k, \\ v(T) - \frac{1}{2}\sum_{j\in T} v(\{j\}), & \text{otherwise.} \end{cases}$$

$$v^{s_k,2}(T) = \begin{cases} v(T), & \text{if } T = S_k, \\ \frac{1}{2}\sum_{j\in T} v(\{j\}), & \text{otherwise.} \end{cases}$$

$$v^3(T) = \frac{1}{2}\sum_{j\in T} v(\{j\}), \quad \text{for all } T \subseteq N.$$

Clearly, $v = v^{S_k,1} + v^{S_k,2}$, $v^{S_k,1}(\{j\}) = v^{S_k,2}(\{j\}) = \frac{1}{2}v(\{j\})$ for all $j \in N$, $v^{S_k,1}(N) = v(N) - \frac{1}{2}\sum_{j\in N} v(\{j\})$ (since $S_k \neq N$), and every player $i \in N\backslash S_k$ is dummifying in $(N, v^{S_k,2})$. By weak coalitional surplus equivalence, $\psi_i(N,v) = \psi_i(N, v^{S_k,1}) + \frac{1}{2}v(\{i\})$ for all $i \in N\backslash S_k$. Moreover, $d(v^{S_k,1} + v^3) = h$, and every player $i \in N$ is dummifying in (N, v^3). Thus, weak coalitional surplus equivalence and the induction hypothesis imply that $\psi_i(N, v^{S_k,1} + v^3) = \psi_i(N, v^{S_k,1}) + v^3(\{i\}) = \psi_i(N, v^{S_k,1}) + \frac{1}{2}v(\{i\}) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v^{S_k,1}(N) + v^3(N)] = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N)$ for all $i \in N$. It follows that $\psi_i(N, v^{S_k,1}) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - \frac{1}{2}v(\{i\})$ for all $i \in N$.

Therefore, $\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) = PD_i(N,v)$ for all $i \in N\backslash S_k$. Since there exists such a S_k for all $i \in N\backslash S$, we obtain $\psi_i(N,v) = PD_i(N,v)$ for all $i \in N\backslash S$.

Case (b): $i \in S$. If $S = \{i\}$, we obtain, by efficiency of ψ and PD and case (a), $\psi_i(N,v) = PD_i(N,v)$. If $|S| \geq 2$, each player $j \in S$ is a member of all coalitions in $\mathcal{S}$. We consider the game (N,v^0) as defined by (2.17). Clearly, all players $i,j \in S$ are weak symmetric in (N,v^0), and thus by proportional-balanced treatment, $\frac{\psi_i(N,v^0)}{v(\{i\})} = \frac{\psi_j(N,v^0)}{v(\{j\})}$ for all $i,j \in S$. Since $v = v^0 + (v - v^0)$ and all players are dummifying in $(N, v - v^0)$, and thus by weak coalitional surplus equivalence, $\psi_j(N,v) = \psi_j(N,v^0) + \frac{v(\{j\})}{2}$ for all $j \in S$. Hence, $\sum_{j\in S} \psi_j(N,v) = \sum_{j\in S}(\psi_j(N,v^0) + \frac{v(\{j\})}{2}) = \sum_{j\in S} \frac{v(\{j\})}{v(\{i\})} \psi_i(N,v^0) + \frac{\sum_{j\in S} v(\{j\})}{2}$ for any $i \in S$. On the other hand, by efficiency and Case (a), $\sum_{j\in S} \psi_j(N,v) = v(N) - \sum_{j\in N\backslash S} \psi_j(N,v) = v(N) - \sum_{j\in N\backslash S} PD_i(N,v) = \frac{\sum_{j\in S} v(\{j\})}{\sum_{k\in N} v(\{k\})} v(N)$. Therefore, $\sum_{j\in S} \frac{v(\{j\})}{v(\{i\})} \psi_i(N,v^0) + \frac{\sum_{j\in S} v(\{j\})}{2} = \frac{\sum_{j\in S} v(\{j\})}{\sum_{k\in N} v(\{k\})} v(N)$. Since $\sum_{j\in S} v(\{j\}) \neq 0$, then $\psi_i(N,v^0) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) - \frac{v(\{i\})}{2}$, and thus $\psi_i(N,v) = PD_i(N,v)$ for all $i \in S$.

The proof of (i) is complete.

(ii) Since it is obvious that the PD value satisfies efficiency and proportional-balanced treatment, we only show that the PD value satisfies weak coalitional surplus monotonicity. Clearly, $w(\{j\}) = cv(\{j\})$ for all $j \in N$ and $v(S) - \sum_{j\in S} v(\{j\}) \geq w(S) - \sum_{j\in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$, imply that $v(N) \geq w(N) - \sum_{j\in N} w(\{j\}) + \sum_{j\in N} v(\{j\}) = w(N) - \sum_{j\in N} w(\{j\}) + \sum_{j\in N} \frac{w(\{j\})}{c} = w(N) - (1 - \frac{1}{c}) \sum_{j\in N} w(\{j\})$. Thus, we obtain $PD_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} v(N) \geq \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} [w(N) - (1 - \frac{1}{c}) \sum_{j\in N} w(\{j\})] = PD_i(N,w) - w(\{i\}) + \frac{1}{c} w(\{i\}) = PD_i(N,w) - w(\{i\}) + v(\{i\})$.

Uniqueness follows from Theorem 2.4(i) and Lemma 2.1. □

Proof of Lemma 2.2. Let $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ be two games such that $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ is dummifying in (N,w). The dummifying player property implies that $\psi_i(N,w) = w(\{i\})$. Then weak additivity implies that $\psi_i(N, v + w) = \psi_i(N,v) + \psi_i(N,w) = \psi_i(N,v) + w(\{i\})$, as desired. □

Proof of Theorem 2.5. We already know that the PD value satisfies efficiency and weak proportional-balanced treatment. Weak coalitional monotonicity is satisfied since $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, implies that $v(N) \geq w(N)$ and thus $PD_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} v(N) \geq \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} w(N) = PD_i(N,w)$. To show uniqueness, suppose that ψ is a value satisfying the three axioms. For any game $(N,v) \in \mathcal{G}_{nz}^N$, define the game (N,w) by

$$w(S) = \begin{cases} v(N), & \text{if } S = N, \\ v(\{j\}), & \text{if } S = \{j\} \text{ for all } j \in N, \\ \min_{T \subseteq N, |T| \geq 2} v(T), & \text{otherwise.} \end{cases}$$

Efficiency and weak proportional-balanced treatment together imply that $\psi_i(N,w) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$ for all $i \in N$. Pick any $i \in N$. Since $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, weak coalitional monotonicity implies that $\psi_i(N,v) \geq \psi_i(N,w) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$. Efficiency then implies that $\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$ for all $i \in N$. □

Proof of Proposition 2.2. For every $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$ and any $j \in N$, $(N \backslash \{j\}, v^x) \in \mathcal{G}_{nz}$.[9] For $x = PD(N,v)$ and $i \in N \backslash \{j\}$, we have

$$\begin{aligned} PD_i(N \backslash \{j\}, v^x) &= \frac{v^x(\{i\})}{\sum_{k \in N \backslash \{j\}} v^x(\{k\})} v^x(N \backslash \{j\}) \\ &= \frac{v(\{i\})}{\sum_{k \in N \backslash \{j\}} v(\{k\})} [v(N) - PD_j(N,v)] \\ &= \frac{v(\{i\})}{\sum_{k \in N \backslash \{j\}} v(\{k\})} [v(N) - \frac{v(\{j\})}{\sum_{k \in N} v(\{k\})} v(N)] \\ &= \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} v(N) \\ &= PD_i(N,v). \end{aligned}$$

□

Remark 2.8. If the definition of projection consistency (see Definition 2.3) is applied on $\mathcal{G}_{nz}$ with $|N| \geq 2$, then Proposition 2.2 is valid on $\mathcal{G}_{nz}$. The proof of the case $|N| = 2$ is given as footnote 9.

Proof of Theorem 2.6. It is straightforward to show that the PD value satisfies proportional standardness. Projection consistency follows from Proposition 2.2. To show the 'only if' part, suppose that ψ is a value satisfying proportional standardness and projection consistency.

If $|N| = 2$, then $\psi(N,v) = PD(N,v)$ follows from proportional standardness.

Proceeding by induction, for $|N| \geq 3$, suppose that $\psi(N',w) = PD(N',w)$ whenever $|N'| = |N| - 1$. Take any $i, j \in N$ such that $i \neq j$. Let $x = \psi(N,v)$ and

[9] Notice that, if $(N,v) \in \mathcal{G}_{nz}$ with $|N| = 2$ and $v(N) = 0$, then for $x = PD(N,v)$, we have that $x_i = x_j = 0$, and thus $(N \backslash \{j\}, v^x) \notin \mathcal{G}_{nz}$ for any $j \in N$. In case $v(N) \neq 0$, for $x = PD(N,v)$, we have $(N \backslash \{j\}, v^x) \in \mathcal{G}_{nz}$, since $[v(\{i\}) > 0 \Rightarrow PD_j(N,v) < v(N) \Rightarrow v^x(\{i\}) > 0]$ (similar if $v(\{i\}) < 0$).

$y = PD(N, v)$. For the two reduced games $(N\backslash\{j\}, v^x)$ and $(N\backslash\{j\}, v^y)$, by the induction hypothesis, we have

$$\begin{aligned} x_i - y_i &= \psi_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y) \\ &= PD_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y). \end{aligned} \tag{2.18}$$

By definition of the PD value and the projection reduced game, we have

$$\begin{aligned} &PD_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y) \\ =&\frac{v^x(\{i\})}{\sum_{k\in N\backslash\{j\}} v^x(\{k\})}(v(N) - x_j) - \frac{v^y(\{i\})}{\sum_{k\in N\backslash\{j\}} v^y(\{k\})}(v(N) - y_j) \\ =&\frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}(y_j - x_j). \end{aligned}$$

Together with (2.18), this implies that, for all $i, j \in N$ with $i \neq j$,

$$x_i - y_i = \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}(y_j - x_j). \tag{2.19}$$

Summing (2.19) over all $i \in N\backslash\{j\}$ yields

$$\sum_{i\in N\backslash\{j\}} (x_i - y_i) = \frac{\sum_{i\in N\backslash\{j\}} v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}(y_j - x_j) = y_j - x_j. \tag{2.20}$$

On the other hand, (2.19) can be written as $v(\{i\})(y_j - x_j) = \sum_{k\in N\backslash\{j\}} v(\{k\})(x_i - y_i)$. Summing this equality over all $j \in N\backslash\{i\}$, we have

$$\begin{aligned} &\sum_{j\in N\backslash\{i\}} \sum_{k\in N\backslash\{j\}} v(\{k\})(x_i - y_i) = \sum_{j\in N\backslash\{i\}} v(\{i\})(y_j - x_j) \\ \Leftrightarrow &\sum_{j\in N\backslash\{i\}} \Big(v(\{i\})(x_i - y_i) + \sum_{k\in N\backslash\{i,j\}} v(\{k\})(x_i - y_i)\Big) = v(\{i\}) \sum_{j\in N\backslash\{i\}} (y_j - x_j) \\ \Leftrightarrow &[(|N| - 1)v(\{i\}) + (|N| - 2) \sum_{j\in N\backslash\{i\}} v(\{j\})](x_i - y_i) = v(\{i\}) \sum_{j\in N\backslash\{i\}} (y_j - x_j). \end{aligned} \tag{2.21}$$

Together with (2.20) and (2.21), it holds that $(|N| - 2)(x_i - y_i) \sum_{j\in N} v(\{j\}) = 0$. Thus, $x_i - y_i = 0$ for all $i \in N$. This shows that $\psi(N, v) = PD(N, v)$. □

Proof of Proposition 2.3. It is obvious that PD satisfies grand worth additivity and the inessential game property for two-player games. To show uniqueness, suppose that ψ is a value on $\mathcal{G}^2_{nz\mathbb{Q}}$ satisfying the two axioms. Let $(N, v) \in \mathcal{G}^2_{nz\mathbb{Q}}$ be an arbitrary game with $N = \{i, j\}$. For any $\alpha \in \mathbb{Q}$, let the game (N, v^α) be defined by $v^\alpha(\{i\}) = v(\{i\})$, $v^\alpha(\{j\}) = v(\{j\})$ and $v^\alpha(N) = \alpha v(N)$. Clearly, $(N, v^\alpha) \in \mathcal{G}^2_{nz\mathbb{Q}}$.

If $\alpha = 0$ then grand worth additivity implies that $\psi(N, v^{\alpha}) = \mathbf{0}$. For any $\alpha \in \mathbb{Z}_{+}$, since $(N, v^{\alpha}) = (N, v^{\alpha-1} \oplus v) = \cdots = (N, \underbrace{v \oplus \cdots \oplus v}_{\alpha})$, grand worth additivity implies $\psi(N, v^{\alpha}) = \alpha\psi(N, v)$. For any $\alpha \in \mathbb{Z}_{-}$, since $(N, v^{\alpha} \oplus \underbrace{v \oplus \cdots \oplus v}_{|\alpha|}) = (N, v^{0})$, grand worth additivity and $\psi(N, v^{0}) = \mathbf{0}$ (from above) imply $\psi(N, v^{\alpha}) = -|\alpha|\psi(N, v) + \psi(N, v^{0}) = \alpha\psi(N, v)$. Similarly, considering (N, v), for any $\alpha \in \mathbb{Z}_{+}$, $(N, v) = (N, \underbrace{v^{\frac{1}{\alpha}} \oplus \cdots \oplus v^{\frac{1}{\alpha}}}_{\alpha})$ implies that $\psi(N, v) = \alpha\psi(N, v^{\frac{1}{\alpha}})$; for any $\alpha \in \mathbb{Z}_{-}$, $(N, v \oplus \underbrace{v^{\frac{1}{\alpha}} \oplus \cdots \oplus v^{\frac{1}{\alpha}}}_{|\alpha|}) = (N, v^{0})$ implies that $\psi(N, v) = \alpha\psi(N, v^{\frac{1}{\alpha}})$.

Next, take any $\alpha \in \mathbb{Q}$ and consider the game (N, v^{α}). Since any rational number can be written as a fraction, we suppose that $\alpha = \frac{k}{m}$ with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}\backslash\{0\}$. Therefore,

$$\psi(N, v^{\alpha}) = \psi(N, v^{\frac{k}{m}}) = k\psi(N, v^{\frac{1}{m}}) = \frac{k}{m}\psi(N, v) = \alpha\psi(N, v).$$

Take any game $(N, v) \in \mathcal{G}^{2}_{nz\mathbb{Q}}$. Taking $\alpha = \frac{v(\{i\})+v(\{j\})}{v(N)}$, (N, v^{α}) is an inessential game, and thus by the inessential game property for two-player games, $\psi_i(N, v^{\alpha}) = v^{\alpha}(\{i\}) = v(\{i\})$. Since $\psi(N, v^{\alpha}) = \alpha\psi(N, v)$, we have

$$\psi_i(N, v) = \frac{1}{\alpha}\psi_i(N, v^{\alpha}) = \frac{v(N)v(\{i\})}{v(\{i\}) + v(\{j\})}.$$

□

Proof of Theorem 2.7. It is clear that PD satisfies the three axioms. To show uniqueness, suppose that ψ is a value on $\mathcal{G}^{2}_{nz}$ satisfying the three axioms. From Proposition 2.3, we already know $\psi(N, v) = PD(N, v)$ for all $(N, v) \in \mathcal{G}^{2}_{nz\mathbb{Q}}$. Now, take any game $(N, v) \in \mathcal{G}^{2}_{nz}$, and let $\{(N, v_m)\}$ be a sequence of games in the class $\mathcal{G}^{2}_{nz\mathbb{Q}}$ such that $\lim_{m\to\infty}(N, v_m) = (N, v)$. Using continuity for two-player games, we have

$$\psi(N, v) = \lim_{(N,v_m)\to(N,v)} \psi(N, v_m) = \lim_{(N,v_m)\to(N,v)} PD(N, v_m) = PD(N, v),$$

where the last equality holds since $PD(N, v)$ is a continuous function with respect to $(N, v) \in \mathcal{G}^{2}_{nz}$. □

2.6 Conclusion

In this chapter, we have provided characterizations of the PD value for TU-games using axioms, such as *proportional-balanced treatment*, *monotonicity*, and *consistency*. It is worth noting that proportional-balanced treatment, in some sense, reflects not only *equal treatment of equals* but also *unequal treatment of unequals*. This axiom captures this feature of the PD value. For games with at least three players, our axiomatic characterizations are similar to the characterizations of the ED value due to van den Brink (2007) and van den Brink and Funaki (2009), and the characterizations of the ESD value due to Casajus and Huettner (2014a). That is, most of them are obtained by weakening one axiom while strengthening another axiom. This shows that the PD value is axiomatically related to these two equal surplus sharing values.

The PD value is of interest for at least two reasons. First, the proportionality principle is often considered as intuitive in various applications of TU-games. Especially, it is desirable to have the option of treating players differently to reflect endogenous characteristics. Second, proportional division methods are often employed in a lot of applications such as bankruptcy problems, claims problems, cost allocation problems and so on.

The PD value depends only on the worths of one-person coalitions and the grand coalition, but ignores the worths of any other intermediate coalitions. In Chapter 4, we will show that the PD value, as well as a family of egalitarian values, is exactly picked out from a larger family of values by imposing projection consistency. This shows a merit of the PD value since the consistency principle is one of the fairness criteria that are widely accepted notions in TU-games.

In the future, we will study other characterizations of the PD value based on some existing characterizations of the ED value as well as the ESD value. Recall that the combination of the PD value and the ED value for joint venture situations is characterized by Moulin (1987). This motivates future research on characterizations of the combination of the PD value and the ED value (or the ESD value) for general TU-games.

Chapter 3

Balanced Externalities and the Proportional Allocation of Nonseparable Contributions

3.1 Introduction

TU-games are applied in many profit and cost allocation problems in economics and operations research. An example is the queueing problem. A *queueing problem* describes a situation where jobs need to be served on a machine one at a time. A queue is efficient if jobs are served in a non-increasing order of their urgency indices. But then the question is how jobs that are served later should be compensated for waiting in the queue. One of the most popular solutions for such queueing problems is the *minimal transfer rule* (Maniquet, 2003) which is obtained by applying the Shapley value to an associated game. Queueing games are so-called *2-additive games* [1], or shortly *2-games*, meaning that the worth is fully generated by coalitions of size two. For a nonnegative 2-game, it is known that the Shapley value, and thus the minimal transfer rule, coincides with several other solutions such as the *nucleolus* (Schmeidler, 1969) or *τ-value* (Tijs, 1987) of the associated queueing game (van den Nouweland et al., 1996).

Solutions for TU-games are usually supported by axiomatizations. In van den Brink and Chun (2012), the minimal transfer rule is axiomatized by *efficiency*, *Pareto indifference*, and *balanced cost reduction*. Whereas efficiency and Pareto indifference are very common axioms, balanced cost reduction requires that the payoff of any player is equal to the total externality she inflicts on the other players with her presence, i.e. a player's payoff equals the sum of all changes in the payoffs of all other players if that player leaves the queueing problem.

In this chapter, which is based on van den Brink et al. (2021), we study the implications of extending this balanced cost reduction property to general TU-games.

[1] Other examples of 2-additive games are the broadcasting games of Bergantiños and Moreno-Ternero (2020a), Bergantiños and Moreno-Ternero (2020b), and Bergantiños and Moreno-Ternero (2021) or the telecommunication games of van den Nouweland et al. (1996).

First, considering the class of *2-games*, we show that the Shapley value (and thus prenucleolus, τ-value) is the unique efficient solution that satisfies *balanced externalities* being a direct translation of balanced cost reduction, requiring that the payoff of any player is equal to the total externalities she inflicts on the other players. Second, we extend this axiom to k-games being games where every worth is generated by coalitions of size k, and obtain a characterization of the Shapley value for k-games. Third, it turns out that this axiom is incompatible with efficiency for general TU-games.

Keeping as close as possible to the idea of having an efficient solution which allocates the worth of the grand coalition in a way that balances a player's payoff with the externalities she inflicts on the other players, we weaken balanced externalities by requiring that every player's payoff is the same *fraction* of her total externality inflicted on the other players. This brings in one extra parameter (the fraction of total externality that is attributed to the players), which makes this weak balanced externalities axiom compatible with efficiency. We show that the unique efficient solution that satisfies this weak balanced externalities axiom is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates the payoffs proportional to the *separable costs* (Moulin, 1985) of the players. It is interesting to note that this value is closely related to the *Separable Costs Remaining Benefits* (SCRB) method (Young et al., 1982) and *Alternative Cost Avoided* (ACA) method (Straffin and Heaney, 1981; Otten, 1993) in cost allocation problems. The SCRB method is commonly used in practice, for example in allocating the costs of multi-purpose water development projects (Straffin and Heaney, 1981; Young et al., 1982).

We also consider the dual value of the PANSC value, being the PD value studied in Chapter 2, which allocates the worth of the grand coalition proportional to the stand-alone worths of the players, and extend weak balanced externalities and the axiomatization mentioned above using mollifier games (i.e. affine combinations of a game and its dual game, see Charnes et al. (1978)). A comparison between the PANSC value and PD value in terms of optimizing satisfaction criteria and associated consistency is given in Li et al. (2020). Finally, we discuss a reduced game consistency property of the PANSC value, which, by duality, follows from the reduced game consistency property of the PD value.

This chapter is organized as follows. After recalling definitions and notation in Section 3.2, in Section 3.3 we extend the axiomatization of the Shapley value for queueing problems and provide an axiomatization by efficiency and balanced externalities for the classes of 2-games, and more general k-games. In Section 3.4, we extend this axiomatization to general TU-games, and show incompatibility of efficiency and balanced externalities. We weaken balanced externalities to get compatibility, and use this weaker axiom to characterize the PANSC value. In Section 3.5, we consider the dual value of the PANSC value, i.e. the PD value. In Section 3.6, we provide other characterizations of the PANSC value. In Section 3.7, we introduce cost allocation problems and compare the PANSC value with cost allocation methods from the literature. The proofs are provided in Section 3.8. Section 3.9 concludes.

3.2 Definitions and notation

We recall some definitions from Chapter 1 that are used in this chapter. Recall that $\mathcal{G}^N$ denotes the class of all games with player set N, and $\mathcal{G}$ denotes the class of all games. For every $T \subseteq N$, $T \neq \emptyset$, the *unanimity* game $(N, u_T) \in \mathcal{G}$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that for every game (N, v), there exists unique weights $\Delta_v(T) \in \mathbb{R}$, $\emptyset \neq T \subseteq N$, such that $v = \sum_{T \subseteq N} \Delta_v(T) u_T$. The weights $\Delta_v(T)$, $\emptyset \neq T \subseteq N$, are the (Harsanyi) *dividends* (Harsanyi, 1959) of the coalitions in game (N, v) and are given by $\Delta_v(T) = v(T)$ if $|T| = 1$, and $\Delta_v(T) = v(T) - \sum_{S \subset T, S \neq T} \Delta_v(S)$ if $|T| \geq 2$.

The *Shapley value* (Shapley, 1953a) is given by

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in N.$$

The *equal allocation of nonseparable cost* (EANSC) value (Moulin, 1985) is given by

$$EANSC_i(N, v) = SC_i(N, v) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} SC_j(N, v)\right) \quad \text{for all } i \in N,$$

where $SC_i(N, v) = v(N) - v(N \backslash \{i\})$ is the *separable cost* of agent $i \in N$ in game (N, v).

3.3 Balanced externalities and 2-games

A game (N, v) is a *2-additive game*, or shortly a *2-game*, if $v(S) = 0$ for all $S \subseteq N$ with $|S| \leq 1$, and $v(S) = \sum_{\substack{T \subseteq S \\ |T|=2}} v(T)$ otherwise. Equivalently, a game (N, v) is a 2-game if and only if only coalitions of size two can have a nonzero dividend, i.e. $\Delta_v(S) \neq 0$ implies that $|S| = 2$. Therefore, in 2-games, all the worth is generated by coalitions of size two. It is known that for nonnegative 2-games (i.e. 2-games in which all worths are nonnegative), the Shapley value coincides with several other values such as the nucleolus and the τ-value (van den Nouweland et al., 1996). For 2-games on N with $|N| \geq 2$, this value is given by $Sh_i(N, v) = \frac{1}{2}(v(N) - v(N \backslash \{i\}))$ for all $i \in N$.

A 2-game can be generalized to a *k-additive game*, or shortly a *k-game*. A game (N, v) is a k-game, if $v(S) = 0$ for all $S \subseteq N$ with $|S| < k$, and $v(S) = \sum_{\substack{T \subseteq S \\ |T|=k}} v(T)$ otherwise. Equivalently, a game (N, v) is a k-game if and only if only coalitions of size k can have a nonzero dividend, i.e. $\Delta_v(S) \neq 0$ implies that $|S| = k$. It is known that for k-games on N with $|N| \geq k$, the Shapley value is given by $Sh_i(N, v) = \frac{1}{k}(v(N) - v(N \backslash \{i\}))$ for all $i \in N$ (van den Nouweland et al., 1996). Also for nonnegative k-games, the Shapley value coincides with the τ-value, but for $k > 2$, the payoff vector assigned to a game by the nucleolus need not coincide with the payoff vector assigned by the Shapley value. In this paper, we do not restrict the sign of the worths of coalitions in 2-games, as well as k-games.

As mentioned in the introduction, queueing games form a proper subset of the class of 2-games. One of the most famous solutions for queueing problems is the minimal transfer rule that is obtained as the Shapley value of an associated queueing game (Maniquet, 2003). In van den Brink and Chun (2012), the minimal tranfer rule is characterized as the unique solution for queueing problems that satisfies efficiency, Pareto indifference, and balanced cost reduction. The question that we address in this chapter is which solutions for games satisfy, or are characterized by, (an extension of) these axioms for general games. We first consider the class of 2-games. Throughout the sequel, we denote by $(N\backslash\{h\}, v^{-h})$ the restricted game on $N\backslash\{h\}$, given by $v^{-h}(S) = v(S)$ for all $S \subseteq N\backslash\{h\}$.

A direct translation of balanced cost reduction for 2-games gives the following property.

- **Balanced externalities**. For every 2-game (N, v) with $|N| \geq 2$, and $h \in N$, it holds that

$$\psi_h(N, v) = \sum_{i \in N\backslash\{h\}} \Big(\psi_i(N, v) - \psi_i(N\backslash\{h\}, v^{-h}) \Big).$$

Note that this axiom is well-defined since $(N\backslash\{h\}, v^{-h})$ is a 2-game if (N, v) is a 2-game. Together with efficiency, this axiom characterizes the Shapley value on the class of 2-games.[2]

Theorem 3.1. *For 2-games, the Shapley value is the unique value that satisfies efficiency and balanced externalities.*

All proofs of results in this chapter are given in Section 3.8.

As a corollary, we have that, on the class of 2-games, the pre-nucleolus and the τ-value are also characterized by efficiency and balanced externalities.

We can generalize this result straightforwardly to the class of k-games by introducing a generalization of balanced externalities for k-games, $k \geq 2$.

- **k-balanced externalities**. For every k-game (N, v) and $h \in N$, it holds that

$$\psi_h(N, v) = \frac{1}{k-1} \sum_{i \in N\backslash\{h\}} \Big(\psi_i(N, v) - \psi_i(N\backslash\{h\}, v^{-h}) \Big).$$

Note again that this axiom is well-defined since $(N\backslash\{h\}, v^{-h})$ is a k-game if (N, v) is a k-game. Together with efficiency and symmetry, this axiom characterizes the Shapley value on the class of k-games.[3]

[2] We can even prove the uniqueness with the weaker axiom of 2-efficiency, requiring efficiency only for games with at most two players.

[3] Similar as for 2-games, it is sufficient to require k-efficiency which requires efficiency only for games with at most k players.

- **Symmetry**. For all $(N, v) \in \mathcal{C} \subseteq \mathcal{G}$ and $i, j \in N$ such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, it holds that $\psi_i(N, v) = \psi_j(N, v)$.

Theorem 3.2. *For k-games, the Shapley value is the unique value that satisfies efficiency, symmetry, and k-balanced externalities.*

Although the proof is almost the same as that of Theorem 3.1, for completeness, it is given in Section 3.8. Note that Theorem 3.1 is a special case of Theorem 3.2 by taking $k = 2$. In this case, symmetry is superfluous. However, for $k \geq 3$, symmetry cannot be taken out since, k-balanced externalities has no bite if $|N| < k$ in which case the game is a null game where the worths of all coalitions equal zero.

3.4 Balanced externalities and the PANSC value for general TU-games

Translating the idea of balanced externalities to general TU-games, we first consider the axiom which requires that the payoff of any player is equal to the total externality she inflicts on the other players with her presence. We call a class of games $\mathcal{C} \subseteq \mathcal{G}$ *subgame closed* if $(N \backslash \{h\}, v^{-h}) \in \mathcal{C}$ for all $(N, v) \in \mathcal{C}$ and $h \in N$.

- **Balanced externalities**. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games. For all $(N, v) \in \mathcal{C}$ with $|N| \geq 2$, and $h \in N$, it holds that

$$\psi_h(N, v) = \sum_{i \in N \backslash \{h\}} \left(\psi_i(N, v) - \psi_i(N \backslash \{h\}, v^{-h}) \right).$$

We investigate the implications of this axiom in the context of games. As it turns out, this axiom is incompatible with efficiency.

Proposition 3.1. *There is no value on $\mathcal{G}$ that satisfies efficiency and balanced externalities.*

Next, we explore whether we can characterize (subclasses of) the class of 2-games as those classes of TU-games where the Shapley value is characterized by efficiency and balanced externalities.

Proposition 3.2. *Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games that contains at least one game (N, v) with $|N| \geq 3$. Then the Shapley value is the unique value that satisfies efficiency and balanced externalities on $\mathcal{C}$ if and only if $\mathcal{C}$ is a subclass of the class of 2-games.*

Note that for $|N| = 2$, every game with $v(\{i\}) = v(\{j\}) = 0$ is a 2-game.

Keeping as close as possible to the idea of having an efficient value which allocates the payoffs of the players to somehow 'balance' the externalities inflicted on the other players, we weaken balanced externalities by requiring that every player's payoff is the same *fraction* of her total externality inflicted on the other players.

- **Weak balanced externalities**. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games. There exists $\alpha \in \mathbb{R}$ such that for all $(N, v) \in \mathcal{C}$ with $|N| \geq 2$ and all $h \in N$, if $\sum_{i \in N \setminus \{h\}} \left(\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})\right) \neq 0$, then

$$\psi_h(N, v) = \alpha \sum_{i \in N \setminus \{h\}} \left(\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})\right). \tag{3.1}$$

Notice that we require the balanced externalities condition to hold only in the case that $\sum_{i \in N \setminus \{h\}} \left(\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})\right) \neq 0$ since otherwise the payoff of h must be zero and is not dependent on α anymore. In the extreme case where equality would hold for all players, then all payoffs would be zero, which would be incompatible with efficiency if $v(N) > 0$.[4]

This weakening of balanced externalities brings in one extra parameter (the fraction of total externality that is attributed to the players), which makes this weak balanced externalities compatible with efficiency. It turns out that the unique efficient solution that satisfies this axiom is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates the worth of the grand coalition proportional to the separable costs of the players. This value coincides with the *proportional repartition of the non-marginal costs* value (Lemaire, 1984) in cost allocation problems.

Let $\mathcal{G}_{sc+} = \{(N, v) \in \mathcal{G} \mid SC_i(N, v) > 0 \text{ for all } i \in N\}$ be the class of games with player set N where all players have positive separable cost. Let $\mathcal{G}^2_{sc+} = \{(N, v) \in \mathcal{G}_{sc+} \mid |N| = 2\}$ and $\mathcal{G}^{\geq 2}_{sc+} = \{(N, v) \mid (N, v) \in \mathcal{G}_{sc+}, |N| \geq 2\}$. We remark that the class $\mathcal{G}^{\geq 2}_{sc+}$ contains the *almost diminishing marginal contributions* games (Leng et al., 2021) with positive stand-alone worths.

Definition 3.1. The *proportional allocation of nonseparable contribution* (PANSC) value on the class $\mathcal{G}^{\geq 2}_{sc+}$ assigns to every game $(N, v) \in \mathcal{G}^{\geq 2}_{sc+}$, the payoff vector

$$PANSC_i(N, v) = \frac{SC_i(N, v)}{\sum_{j \in N} SC_j(N, v)} v(N) \quad \text{for all } i \in N.$$

We restrict ourselves to the class $\mathcal{G}^{\geq 2}_{sc+}$ in order to avoid dividing by a zero denominator.

Note that equivalently the PANSC value first assigns to every player its separable cost and allocates the remainder (the total nonseparable cost) proportional to the separable costs. Thus, the difference from the EANSC value (mentioned in Section 3.2) is that, after each player getting its separable cost, the PANSC allocates the total nonseperable cost proportional to the separable costs, while that EANSC allocates it equally over all players. We provide a further comparison between the PANSC and EANSC value in Section 3.7.

Theorem 3.3. *The PANSC value is the unique value on $\mathcal{G}^{\geq 2}_{sc+}$ that satisfies efficiency and weak balanced externalities.*

[4] This would occur if $v(N) = v(N \setminus \{h\}) > 0$ for all $h \in N$.

The PANSC value can be seen as a multiplicative normalization of the separable costs to allocate $v(N)$. In the literature, this proportional allocation of nonseparable cost (or contribution) is less popular than the additive normalization of the nonseparable cost (or contribution), as done by the more famous EANSC value. However, as we will see in Section 3.7, for some cases the PANSC value coincides with well-known and applied solutions in cost allocation. In this section, we promoted the PANSC value by the weak balanced externalities axiom, which is inspired by a notion of fairness in queueing problems.

3.5 The dual value: proportional division

Every value has its dual value which, instead of focusing on what coalitions can earn, considers what happens if any coalition leaves assuming that the grand coalition is already formed. In some cases, the dual value equals the value itself, in which case we call this value *self-dual*. An example of a self-dual value is the Shapley value.

Formally, the *dual* of game $(N, v) \in \mathcal{G}$ is the game $(N, v^*) \in \mathcal{G}$ given by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$. The *dual* of value ψ is the value ψ^* that assigns to every game the payoff vector that ψ assigns to the dual game, i.e. $\psi^*(N, v) = \psi(N, v^*)$ for all $(N, v) \in \mathcal{G}$. The dual of the PANSC value is the PD value, extensively discussed in Chapter 2, which allocates $v(N)$ proportional to the stand-alone worths of the players. Recall from Chapter 2 that the *PD value* on the class of games with positive stand-alone worths is given by

$$PD_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) \quad \text{for all } (N, v) \in \mathcal{G}_{nz+} \text{ and } i \in N,$$

where $\mathcal{G}_{nz+} = \{(N, v) \in \mathcal{G} \mid v(\{i\}) > 0 \text{ for all } i \in N\}$ is the class of games on N where all stand-alone worths are positive.[5]

Proposition 3.3. *For every game $(N, v) \in \mathcal{G}_{nz+}$, it holds that $PANSC(N, v^*) = PD(N, v)$.*

Note that, under efficiency, (3.1) in the definition of weak balanced externalities can be written as

$$\psi_h(N, v) = \alpha \left(\sum_{i \in N \backslash \{h\}} \psi_i(N, v) - v(N \backslash \{h\}) \right)$$

for every $h \in N$ with $\sum_{i \in N \backslash \{h\}} \psi_i(N, v) - v(N \backslash \{h\}) \neq 0$. This shows that the payoff assigned to player i is proportional to the externality she inflicts on all other players, assuming that without player i, the coalition of remaining players earns its worth $v(N \backslash \{h\})$ in the original game. Instead of this worth, we can also consider other possibilities to measure this externality.

[5] This value is also well-defined if all stand-alone worths are negative, as was allowed in Chapter 2.

As an example, consider the class of *mollifier games* in Charnes et al. (1978), that is based on affine combinations of a game and its dual. Formally, for $\beta \in \mathbb{R}$, define the mollifier game (N, v'_β) as follows:

$$v'_\beta(S) = \beta v(S) + (1-\beta)v^*(S) \quad \text{for all } S \subseteq N. \tag{3.2}$$

- **Mollified weak balanced externalities.** Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games and let $\beta \in \mathbb{R}$. There exists $\alpha \in \mathbb{R}$ such that for all $(N, v) \in \mathcal{C}$ with $|N| \geq 2$ and all $h \in N$, if $\sum_{i \in N \setminus \{h\}} \psi_i(N, v) - v'_\beta(N \setminus \{h\}) \neq 0$, then

$$\psi_h(N, v) = \alpha \left(\sum_{i \in N \setminus \{h\}} \psi_i(N, v) - v'_\beta(N \setminus \{h\}) \right).$$

Although the worth $v'_\beta(N \setminus \{i\})$ need not be equal to $v(N \setminus \{i\})$, $i \in N$, it turns out that mollified weak balanced externalities is compatible with efficiency, and characterizes the following value among the efficient values. Defining $\mathcal{G}_{\beta+} = \{(N, v) \in \mathcal{G} \mid \sum_{j \in N} \beta v^*(\{j\}) + (1-\beta)v(\{j\})) > 0 \quad \text{for all } i \in N\}$, the proof follows a similar line as the proof of Theorem 3.3.

Theorem 3.4. *Let $\beta \in \mathbb{R}$. A value ψ on $\mathcal{G}_{\beta+}$ satisfies efficiency and mollified weak balanced externalities if and only if $\psi = \psi^\beta$ with ψ^β given by*

$$\psi_i^\beta(N, v) = \frac{\beta v^*(\{i\}) + (1-\beta)v(\{i\})}{\sum_{k \in N} (\beta v^*(\{k\}) + (1-\beta)v(\{k\}))} v(N) \quad \textit{for all } i \in N.$$

Note that ψ^1 is the PANSC value, and 1-weak balanced externalities is very similar to weak balanced externalities (and, in fact, is equivalent to weak balanced externalities under efficiency). As another special case, ψ^0 coincides with the PD value.

Other worths instead of $v'_\beta(N \setminus \{h\})$ could be used for coalition $N \setminus \{h\}$ in the definition of mollified weak balanced externalities, but there are some restrictions. For example, assuming that there is no impact from player h leaving, and thus the remaining players earn $v(N)$ would give that $\psi_h(N, v) = \alpha \left(\sum_{i \in N \setminus \{h\}} \psi_h(N, v) - v(N) \right) = \alpha(v(N) - \psi_h(N, v) - v(N)) = -\alpha \psi_h(N, v)$, which implies that $\alpha = -1$. Obviously, this is a restatement of efficiency.

3.6 Axiomatic characterizations of the PANSC value

In this section, considering a variable player set, we characterize the PANSC value involving a *reduced game consistency* axiom. Then, we provide characterizations of the PANSC value for two-player games.

3.6.1 Consistency

By duality, the result in this subsection is very closely related to the reduced game consistency result for the PD value in Section 2.3.3.

We consider the following reduced game. If a player $j \in N$ leaves game (N, v) with a certain payoff, then the *complement reduced game* (see Thomson (2011a)), also known as *dual projection consistency game* in van den Brink et al. (2016), is a game on the remaining player set that assigns to every subset of $N \backslash \{j\}$ its worth together with player j in the original game minus the payoff that is assigned to player j. We might consider that player j leaves the game with a fixed payoff, but commits to cooperate with any coalition of remaining players. In return, player j is guaranteed her payoff.

Definition 3.2. Given game $(N, v) \in \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{G}$, with $|N| \geq 2$, player $j \in N$ and payoff vector $x \in \mathbb{R}^N$, the complement reduced game with respect to j and x is the game $(N \backslash \{j\}, v^x)$ given by

$$v^x(S) = \begin{cases} v(S \cup \{j\}) - x_j & \text{for any } S \subseteq N \backslash \{j\}, S \neq \emptyset; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Complement consistency requires that the payoffs assigned to the remaining players in $N \backslash \{j\}$, after player j leaving the game with her payoff according to value ψ, is the same in the reduced game as in the original game.

Definition 3.3. A value ψ on $\mathcal{C} \subseteq \mathcal{G}$ satisfies *complement consistency* if for every game $(N, v) \in \mathcal{C}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N, v)$, it holds that $(N \backslash \{j\}, v^x) \in \mathcal{C}$, and $\psi_i(N, v) = \psi_i(N \backslash \{j\}, v^x)$ for all $i \in N \backslash \{j\}$.

The following result follows straightforwardly from Proposition 2.2 in Subsection 2.3.3. Since their domains are different, we provide the proof for ease of the reader.

Proposition 3.4. *The PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$ satisfies complement consistency.*

Complement consistency is the dual axiom of *projection consistency* (see Definition 2.3) that is used to axiomatize the PD value in Section 2.3.3. Axiomatizations using reduced game consistency usually assume a specific allocation for two-player games. For the PD value, this is proportional standardness which requires that for two-player games (with positive stand-alone worths) the worth of the grand coalition is allocated proportional to the stand-alone worths of the players. The PANSC value obviously satisfies the dual of this standardness. Recall that $\mathcal{G}^2_{sc+} = \{(N, v) \in \mathcal{G}_{sc+} \mid |N| = 2\}$.

- **Dual proportional standardness**. For all $(N, v) \in \mathcal{G}^2_{sc+}$, it holds that

$$\psi_i(N, v) = \frac{v(\{i,j\}) - v(\{j\})}{(v(\{i,j\}) - v(\{i\})) + (v(\{i,j\}) - v(\{j\}))} v(\{i,j\}) \quad \text{for } i \in N = \{i,j\}.$$

Theorem 3.5. *The PANSC value is the unique value on $\mathcal{G}_{sc+}^{\geq 2}$ that satisfies complement consistency and dual proportional standardness.*

Although the proof is almost the same as that of Theorem 2.6 in Subsection 2.3.3, for completeness, it is given in Section 3.8.

3.6.2 Characterizations for two-player games

As mentioned in Subsection 2.3.4, standardness is a quite strong axiom since it coincides with the definition of some value for two-player games. Instead of dual proportional standardness, we could also use the following axiom, which requires that for two-player games where the worth of the grand coalition and the ratio of the separable contributions of both players is equal, their payoffs are equal.

- **Dual proportionality**. For every two games $(N, v), (N, v') \in \mathcal{G}_{sc+}^{2}$ such that (i) $v(N) = v'(N)$ and (ii) there is $\alpha > 0$ such that $SC_i(N, v) = \alpha SC_i(N, v')$ for all $i \in N$, it holds that $\psi(N, v) = \psi(N, v')$.

Dual proportionality and complement consistency are not sufficient to characterize the PANSC value on $\mathcal{G}_{sc+}^{\geq 2}$. Therefore, we additionally require the inessential game property and continuity, but only for two-player games in $\mathcal{G}_{sc+}^{2}$. Mind that the inessential game property in $\mathcal{G}_{nz}^{2}$ is used in characterizing the PD value in Subsection 2.3.4.

- **Inessential game property for two-player games**. For every game $(N, v) \in \mathcal{G}_{sc+}^{2}$ with $N = \{i, j\}, i \neq j$, if $v(\{i\}) + v(\{j\}) = v(\{i, j\})$, it holds that $\psi_i(N, v) = v(\{i\})$ and $\psi_j(N, v) = v(\{j\})$.
- **Continuity for two-player games**. For all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{G}_{sc+}^{2}$ such that $\lim_{k\to\infty}(N, w_k) = (N, v)$, it holds that $\lim_{k\to\infty} \psi(N, w_k) = \psi(N, v)$.

The above three axioms characterize the PANSC value on $\mathcal{G}_{sc+}^{2}$.

Lemma 3.1. *The PANSC value is the unique value on $\mathcal{G}_{sc+}^{2}$ that satisfies dual proportionality, the inessential game property for two-player games, and continuity for two-player games.*

Logical independence of the axioms used in Lemma 3.1 can be shown by the following alternative solutions.

(i) The PD value satisfies all axioms except dual proportionality.

(ii) The value $\psi_i(N, v) = \frac{SC_i(N,v)v(N)}{2\sum_{j\in N} SC_j(N,v)} + \frac{v(N)}{2|N|}$ for all $(N, v) \in \mathcal{G}_{sc+}^{2}$ and $i \in N$, satisfies all axioms except the inessential game property for two-player games.

(iii) The value, for all $(N, v) \in \mathcal{G}^2_{sc+}$ and $i \in N$, given by

$$\psi_i(N,v) = \begin{cases} PANSC_i(N,v) & \text{if } v(N) \neq 0 \\ ESD_i(N,v) & \text{if } v(N) = 0 \end{cases}$$

satisfies all axioms except continuity for two-player games.

Theorem 3.5 and Lemma 3.1 together yield the following characterization of the PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$. The proof is omitted.

Corollary 3.1. The PANSC value is the unique value on $\mathcal{G}^{\geq 2}_{sc+}$ that satisfies dual proportionality, the inessential game property for two-player games, continuity for two-player games, and complement consistency.

Next, we provide alternative characterizations of the PANSC value on $\mathcal{G}^2_{sc+}$.

As a variation of dual proportionality, *dual grand worth proportionality* requires that for two-player games where the separable contributions of each player are the same, the ratio of the payoffs of both players equals the ratio of the worths of the grand coalition of two games.

- **Dual grand worth proportionality**. For every two games $(N, v), (N, v') \in \mathcal{G}^2_{sc+}$ such that (i) $N = \{i, j\}$, $SC_i(N, v) = SC_i(N, v')$, $SC_j(N, v) = SC_j(N, v')$, and (ii) there is $\alpha \in \mathbb{R}$ such that $v(N) = \alpha v'(N)$, it holds that $\psi(N, v) = \alpha \psi(N, v')$.

It turns out that the PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$ is characterized by the inessential game property for two-player games and dual grand worth proportionality. The proof is obvious and is omitted.

Lemma 3.2. *The PANSC value is the unique value on $\mathcal{G}^2_{sc+}$ that satisfies the inessential game property for two-player games and dual grand worth proportionality.*

The PD value satisfies neither complement consistency nor dual proportional standardness but, being the dual value of the PANSC value, is characterized by the dual axioms. Based on this duality, we provide two results that are closely related to Proposition 2.3 and Theorem 2.7.

Denote $\mathcal{G}^2_{sc\mathbb{Q}} = \{(N, v) \in \mathcal{G}^2_{sc+} \mid v(S) \in \mathbb{Q} \text{ for all } S \subseteq N\}$, so the worths of coalitions in games in $\mathcal{G}^2_{sc\mathbb{Q}}$ are rational numbers. The following axiom on $\mathcal{G}^2_{sc\mathbb{Q}}$ is the dual of grand worth additivity introduced in Subection 2.3.4.

- **Relevant additivity**: For two games $(N, v), (N, v') \in \mathcal{G}^2_{sc\mathbb{Q}}$ such that (i) $N = \{i, j\}$, and (ii) there is $a \in \mathbb{Q}$ such that $v'(S) = v(S) + a$ for all $S \subseteq N$ with $S \neq \emptyset$, it holds that

$$\psi(N, v) + \psi(N, v') = \psi(N, v + I_{v'}) = \psi(N, v' + I_v),$$

where the game $(N, v + I_{v'})$ is defined as: $(v + I_{v'})(S) = v(S) + v'(N)$ for all $S \subseteq N$ with $S \neq \emptyset$.

Proposition 3.5. *The PANSC value is the unique value on $\mathcal{G}^2_{sc\mathbb{Q}}$ that satisfies the inessential game property for two-player games and relevant additivity.*

Similar to Theorem 2.7, continuity for two-player games together with the axioms in Proposition 3.5 characterize the PANSC value on $\mathcal{G}^2_{sc+}$. The proof is obvious and is omitted.

Lemma 3.3. *The PANSC value is the unique value on $\mathcal{G}^2_{sc+}$ that satisfies relevant additivity, the inessential game property for two-player games, and continuity for two-player games.*

3.7 Comparison with other values

In this section, we discuss the relationship between the PANSC value and two other existing values, in particular the EANSC value that was mentioned in Section 3.2, and the SCRB method which is popular for cost allocation problems.

3.7.1 Comparison with the EANSC value

The PANSC and EANSC values are both based on the separable costs of the players. Whereas the EANSC value assigns to every player its separable cost and allocates the remainder (the total nonseparable cost) equally over the players, the PANSC value allocates the worth of the grand coalition proportional to the separable costs, which is equivalent to first assigning to every player its separable cost and allocating the remainder (the total nonseparable cost) proportional to the separable costs. The next proposition gives two sufficient conditions for the EANSC and PANSC values giving the same payoff vector.

Proposition 3.6. *For every game $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$, $EANSC(N,v) = PANSC(N,v)$ if and only if $v(N) = \sum_{k \in N} SC_k(N,v)$ or $v(N \backslash \{i\}) = v(N \backslash \{j\})$ for all $i,j \in N$.*

The EANSC value satisfies the well-known standardness due to Hart and Mas-Colell (1989).

- **Standardness**. For all $(N,v) \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$, with $|N| = 2$, it holds that

$$\psi_i(N,v) = v(\{i\}) + \frac{1}{2}[v(\{i,j\}) - v(\{i\}) - v(\{j\})] \quad \text{for } N = \{i,j\}.$$

The EANSC value also satisfies complement consistency. Similar to Theorem 3.5, complement consistency and standardness together characterize the EANSC value on $\mathcal{G}$ (also on $\mathcal{G}^{\geq 2}_{sc+}$). Moulin (1985) considers this fact in cost allocation problems, but we give the exact proof for completeness, see Section 3.8.

Theorem 3.6. *The EANSC value is the unique value on $\mathcal{G}$ that satisfies complement consistency and standardness.*

Similar as the PANSC value (respectively, the EANSC value) is the multiplicative (respectively, additive) normalization of the separable costs, the PD value (respectively, the ESD value) is the multiplicitave (respectively, additive) normalization of the stand-alone worths, as summarized in Table 3.1.

	Multiplicative normalization	Additive normalization
Separable cost $SC_i(N,v)$	PANSC	EANSC
Stand-alone worth $v(\{i\})$	PD	ESD

TABLE 3.1: Individual assignments and normalization

In Section 3.4, we considered a 'multiplicative normalization' of balanced externalities, and saw that it characterizes the PANSC value as the unique efficient value satisfying this axiom. An alternative could be to consider an 'additive normalization'. Combining the two variations gives the following axiom (and thus weaker than both).

- α, γ**-weak balanced externalities**. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games. For all $(N,v) \in \mathcal{C}$ with $|N| \geq 2$, there exist $\alpha, \gamma \in \mathbb{R}$ such that, for every $h \in N$,

$$\psi_h(N,v) = \alpha \sum_{i \in N\backslash\{h\}} \left(\psi_i(N,v) - \psi_i(N\backslash\{h\}, v^{-h})\right) + \gamma.$$

If ψ is efficient, then this axiom implies that

$$\psi_h(N,v) = \alpha(v(N) - \psi_h(N,v) - v(N\backslash\{h\})) + \gamma$$

and thus

$$\psi_h(N,v) = \frac{\alpha SC_h(N,v) + \gamma}{1+\alpha}.$$

Special cases of efficient values are:

(i) If $\alpha = 0$, then we get the ED value, being $ED_h = \frac{v(N)}{|N|}$ for all $h \in N$.

(ii) If $\gamma = 0$, then we get the PANSC value.

(iii) If $\alpha = 1$, then we get a modified EANSC value, being

$$MEANS_h(N,v) = \frac{1}{2}SC_h(N,v) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} \frac{SC_j(N,v)}{2}\right) \text{ for all } h \in N.$$

The MEANSC value coincides with the Shapley value for 2-games, and thus with the minimal transfer rule for queueing games, as it should by the axioms. Notice that the EANSC value does not coincide with the Shapley value for 2-games, and does not belong to this class.

3.7.2 Comparison with the SCRB method

In this subsection, we show that the PANSC value is closely related to the well-known *Separable Costs Remaining Benefits (SCRB) method* in cost allocation problems. Cost allocation problems have obtained much attention in the literature and have been applied to address real problems. One of the most famous empirical examples is the Tennessee Valley Authority (TVA) Act which is designed to assign the cost of TVA projects specifically among the several purposes involved (Ransmeier, 1942). The SCRB method is commonly used in practice for allocating the costs of multi-purpose water development projects (Straffin and Heaney, 1981; Young et al., 1982). It is based on a simple yet appealing idea that joint costs should be allocated in proportion to the willingness to pay of the players. For a survey of such method, we refer to Tijs and Driessen (1986).

A *cost allocation problem* is a triple (N, c, b), where $N \subset \mathbb{N}$ is a set of participants or players, $c : 2^N \to \mathbb{R}$ is a cost function with $c(\emptyset) = 0$, and $b = (b(i))_{i \in N}$ is a profile where $b(i)$ is the benefit to player i if her purposes are served. For any $S \subseteq N$, $c(S)$ is the cost of serving S which is the minimal cost of providing the service to the players in S. The objective is to allocate the total cost $c(N)$ among all players. A *cost allocation method* or *solution* is a function which assigns an allocation vector $x \in \mathbb{R}^N$ to each cost allocation problem (N, c, b).

Notice that the pair (N, c) is mathematically equivalent to a game (N, v). Because of its different interpretation, the literature often speaks about a cost game, respectively a profit game. Solutions can be defined for both types of games. In the literature, one can find solutions that are applicable for cost as well as profit games. Also the PANSC value is applicable in both contexts. In the case of cost games, we speak about $SC_i(N, c) = c(N) - c(N \backslash \{i\})$ as the separable cost of player i in cost game (N, c), and we refer to $NSC(N, c) = c(N) - \sum_{j \in N} SC_j(N, c)$ as the *nonseperable cost*.[6]

Since player i would not be willing to pay more than $\min\{b(i), c(\{i\})\}$ to participate in the joint project, $\min\{b(i), c(\{i\})\} - SC_i(N, c)$ is considered as player i's remaining benefit. The SCRB method assigns to each player her separable cost, and then allocates the nonseparable cost in proportion to the remaining benefits of players. Formally, the SCRB method is given by

$$SCRB_i(N, c, b) = SC_i(N, c) + \frac{\min\{b(i), c(\{i\})\} - SC_i(N, c)}{\sum_{j \in N}(\min\{b(j), c(\{j\})\} - SC_j(N, c))} \cdot NSC(N, c).$$

A variant of the SCRB method, the Alternative Cost Avoided (ACA) method, is studied by Straffin and Heaney (1981) and Otten (1993), and is given by

$$ACA_i(N, c, b) = SC_i(N, c) + \frac{c(\{i\}) - SC_i(N, c)}{\sum_{j \in N}(c(\{j\}) - SC_j(N, c))} \cdot NSC(N, c).$$

[6] Since cost and profit games are mathematically equivalent, we denote the separable costs by $SC_i(N, c)$, and $SC_i(N, v)$ respectively, depending on the context.

It is clear that if $b(j) \geq c(\{j\})$ for all $j \in N$, then the SCRB method coincides with the ACA method, i.e. $SCRB(N,c,b) = ACA(N,c,b)$.

A cost allocation problem (N,c,b) can be transformed into a game (N,v) in two ways. One is using an *anti-game* defined by $v(S) = -c(S)$ for all $S \subseteq N$. This game assigns to every coalition the total cost to provide the service for this coalition (in nonnegative terms). The other is using a *cost saving game* (Young et al., 1982) defined by $v(S) = \sum_{k \in S} c(\{k\}) - c(S)$ for all $S \subseteq N$. This game assigns to every coalition the cost saving it can earn when cooperating and providing the service together for all players in the coalition instead of every player providing the service for herself.

It turns out that in the special cases that the individual benefits are zero, or at least equal to the individual costs, the SCRB method coincides with a variation of the PANSC value to one of the associated games.

Proposition 3.7. *Consider cost allocation problem (N,c,b).*

(i) *If $b(j) = 0$ for all $j \in N$, then $SCRB(N,c,b) = -PANSC(N,v)$, where (N,v) is the associated anti-game.*

(ii) *If $b(j) \geq c(\{j\})$ for all $j \in N$, then $SCRB_j(N,c,b) = c(\{j\}) - PANSC_j(N,v)$ for all $j \in N$, where (N,v) is the associated cost saving game.*

3.8 Proofs

Proof of Theorem 3.1. It is well-known that the Shapley value is efficient. To show that the Shapley value satisfies balanced externalities, let (N,v) be a 2-game such that $|N| \geq 2$, and $h \in N$. Then

$$\begin{aligned}
\sum_{i \in N \setminus \{h\}} \left(Sh_i(N,v) - Sh_i(N \setminus \{h\}, v^{-h}) \right) &= \sum_{i \in N \setminus \{h\}} \left(\sum_{\substack{S \subseteq N, |S|=2 \\ i \in S}} \frac{\Delta_v(S)}{2} - \sum_{\substack{S \subseteq N, |S|=2 \\ i \in S, h \notin S}} \frac{\Delta_v(S)}{2} \right) \\
&= \sum_{i \in N \setminus \{h\}} \sum_{\substack{S \subseteq N, |S|=2 \\ i,h \in S}} \frac{\Delta_v(S)}{2} \\
&= \sum_{\substack{S \subseteq N, |S|=2 \\ h \in S}} \frac{\Delta_v(S)}{2} \\
&= Sh_h(N,v),
\end{aligned}$$

where the first equality follows since $\Delta_v(S) = \Delta_{v^{-h}}(S)$ for all $S \subseteq N \setminus \{h\}$, and the last equality follows since only coalitions of size 2 have a nonzero dividend. This shows that the Shapley value satisfies balanced externalities.

We show the 'only if' part by induction on $|N|$. If $|N| = 1$, then $\psi_i(N,v) = v(\{i\}) = 0 = Sh_i(N,v)$ by efficiency. If $|N| = 2$ such that $N = \{i,j\}$, then balanced externalities implies that $\psi_i(N,v) = \psi_j(N,v) - \psi_j(\{j\}, v^{-i}) = \psi_j(N,v)$. With efficiency and the case $|N| = 1$ above, it then follows that $\psi_i(N,v) = \psi_j(N,v) = \frac{v(N)}{2}$.

We will establish the claim for an arbitrary number of players by an induction argument. As induction hypothesis, suppose that uniqueness holds for all $N' \subset \mathbb{N}$ such that $2 \leq |N'| \leq |N| - 1$. Consider 2-game (N, v). For any $h \in N$, balanced externalities yields

$$\psi_h(N,v) = \sum_{i \in N \setminus \{h\}} \left(\psi_i(N,v) - \psi_i(N \setminus \{h\}, v^{-h}) \right). \tag{3.3}$$

By the induction hypothesis, the $\psi_i(N \setminus \{h\}, v^{-h})$, $i, h \in N$, $i \neq h$, are uniquely determined. Since $|N| \geq 3$, (3.3) and efficiency yield a system of $(|N| - 1) + 1 = |N|$ linearly independent equations in the $|N|$ unkowns $\psi_h(N,v)$, $h \in N$, which thus are uniquely determined. $\square$

Proof of Theorem 3.2. The proof is similar to that of Theorem 3.1, but we put it here for completeness as follows.

It is well-known that the Shapley value satisfies efficiency and symmetry. To show that the Shapley value satisfies k-balanced externalities, consider k-game (N, v) and $h \in N$. Then

$$\begin{aligned}
\sum_{i \in N \setminus \{h\}} \left(Sh_i(N,v) - Sh_i(N \setminus \{h\}, v^{-h}) \right) &= \sum_{i \in N \setminus \{h\}} \left(\sum_{\substack{S \subseteq N, |S| = k \\ i \in S}} \frac{\Delta_v(S)}{k} - \sum_{\substack{S \subseteq N, |S| = k \\ i \in S, h \notin S}} \frac{\Delta_v(S)}{k} \right) \\
&= \sum_{i \in N \setminus \{h\}} \sum_{\substack{S \subseteq N, |S| = k \\ i, h \in S}} \frac{\Delta_v(S)}{k} \\
&= (k-1) \sum_{\substack{S \subseteq N, |S| = k \\ h \in S}} \frac{\Delta_v(S)}{k} \\
&= (k-1) Sh_h(N,v),
\end{aligned}$$

where the first equality follows since $\Delta_v(S) = \Delta_{v^{-h}}(S)$ for all $S \subseteq N \setminus \{h\}$, and the third equality follows since every k-size coalition containing player h appears $k-1$ times in the summation (once for every other player $i \in N \setminus \{h\}$). This shows that the Shapley value satisfies k-balanced externalities.

We show the 'only if' part by induction on $|N|$. Let (N, v) be a k-game such that $k \geq 3$ (the case $k = 2$ is already shown by Theorem 3.1). If $|N| \leq k$, then all players are symmetric in (N, v), and thus symmetry and efficiency imply that ψ and the Shapley value coincide.

As induction hypothesis, suppose that uniqueness holds for all $N' \subset \mathbb{N}$ such that $k \leq |N'| \leq |N| - 1$. For any k-game (N, v) and $h \in N$, k-balanced externalities yields

$$\psi_h(N,v) = \frac{1}{k-1} \sum_{i \in N \setminus \{h\}} \left(\psi_i(N,v) - \psi_i(N \setminus \{h\}, v^{-h}) \right). \tag{3.4}$$

By the induction hypothesis, the $\psi_i(N\backslash\{h\}, v^{-h})$, $i, h \in N$, $i \neq h$, are uniquely determined. Since $|N| \geq 3$, (3.4) and efficiency yield a system of $(|N|-1)+1 = |N|$ linearly independent equations in the $|N|$ unkowns $\psi_h(N, v)$, $h \in N$, which thus are uniquely determined. □

Proof of Proposition 3.1. Consider a three-player game $(N, v) \in \mathcal{G}$ with $N = \{1, 2, 3\}$. Let ψ be a value satisfying efficiency and balanced externalities. Then, we have the following six equations from balanced externalities (the first equality in each line) and efficiency (the second equality in each line):

$$\begin{aligned}
-\psi_1(N,v) + \psi_2(N,v) + \psi_3(N,v) &= \psi_2(N\backslash\{1\}, v^{-1}) + \psi_3(N\backslash\{1\}, v^{-1}) = v(\{2,3\}),\\
\psi_1(N,v) - \psi_2(N,v) + \psi_3(N,v) &= \psi_1(N\backslash\{2\}, v^{-2}) + \psi_3(N\backslash\{2\}, v^{-2}) = v(\{1,3\}),\\
\psi_1(N,v) + \psi_2(N,v) - \psi_3(N,v) &= \psi_1(N\backslash\{3\}, v^{-3}) + \psi_2(N\backslash\{3\}, v^{-3}) = v(\{1,2\}).
\end{aligned}$$

Further, efficiency implies that

$$\psi_1(N,v) + \psi_2(N,v) + \psi_3(N,v) = v(\{1,2,3\}).$$

These four equations can be simplified to

$$\begin{aligned}
\psi_1(N,v) &= \frac{1}{2}(v(\{1,2\}) + v(\{1,3\})),\\
\psi_2(N,v) &= \frac{1}{2}(v(\{1,2\}) + v(\{2,3\})),\\
\psi_3(N,v) &= \frac{1}{2}(v(\{1,3\}) + v(\{2,3\})), \text{ and}\\
\psi_3(N,v) &= \frac{1}{2}(v(\{1,2,3\}) - v(\{1,2\})).
\end{aligned}$$

Looking at the last two equations, this system of equations clearly has only a solution if $v(\{1,3\}) + v(\{2,3\}) = v(\{1,2,3\}) - v(\{1,2\})$, or $v(\{1,2,3\}) = v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})$, which implies that (N, v) is a 2-game. □

Proof of Proposition 3.2. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games that contains at least one game (N, v) with $|N| \geq 3$. For the class of 2-games, the 'if' part follows from Theorem 3.1. For any subgame closed subclass of the class of 2-games, the proof goes in a similar way.

Next, we prove the 'only if' part by induction on $|N|$. Suppose that the Shapley value is the unique solution that satisfies efficiency and balanced externalities on $\mathcal{C}$.

Initialization. For $|N| = 3$, it follows from the proof of Proposition 3.1 that $\mathcal{C}$ should be a subclass of 2-games.

Induction hypothesis. Suppose that the Shapley value being characterized by efficiency and balanced externalities on every class $\mathcal{C} \subseteq \mathcal{G}$ with $|N| \leq d$ ($d \geq 3$) for every $(N, v) \in \mathcal{C}$, implies that $\mathcal{C}$ is a class of 2-games.

Induction step. Consider any game (N,v) such that $|N| = d+1$. We already know that $v(S) = v^{-h}(S) = \sum_{\substack{T\subseteq S\\|T|=2}} v(T)$ for all $S \subseteq N\backslash\{h\}$ and $h \in N$, since $(N\backslash\{h\}, v^{-h})$ is a 2-game by the induction hypothesis. Let ψ be a solution satisfying efficiency and balanced externalities. Then,

$$\begin{aligned}
\psi_h(N,v) &= \sum_{i\in N\backslash\{h\}} \psi_i(N,v) - \sum_{i\in N\backslash\{h\}} \psi_i(N\backslash\{h\}, v^{-h}) \\
&= \sum_{i\in N\backslash\{h\}} \psi_i(N,v) - v^{-h}(N\backslash\{h\}),
\end{aligned}$$

where the first equality follows from balanced externalities and the second equality follows from efficiency.

Summing this equality over all $h \in N$ yields

$$\sum_{h\in N} \psi_h(N,v) = (|N|-1)\sum_{h\in N} \psi_h(N,v) - \sum_{h\in N} v^{-h}(N\backslash\{h\}),$$

so $(|N|-2)\sum_{h\in N} \psi_h(N,v) = \sum_{h\in N} v^{-h}(N\backslash\{h\})$. Efficiency of ψ then implies that

$$(|N|-2)v(N) = \sum_{h\in N} v^{-h}(N\backslash\{h\}) = \sum_{h\in N}\sum_{\substack{S\subseteq N\backslash\{h\}\\|S|=2}} v(S) = (|N|-2)\sum_{\substack{S\subseteq N\\|S|=2}} v(S),$$

where the second equality follows from the induction hypothesis, and the third follows since in $\sum_{S\subseteq N\backslash\{h\},|S|=2} v(S)$, the two-player coalition worth $v(S)$, $S = \{i,j\}$, appears once for each $h \in N\backslash\{i,j\}$. Therefore, we obtain that $v(N) = \sum_{S\subseteq N,|S|=2} v(S)$, which implies that (N,v) must be a 2-game. □

Proof of Theorem 3.3. It is obvious that the PANSC value is efficient. To show that the PANSC value satisfies weak balanced externalities, take any $h \in N$. We consider two cases.

Case (i). If $\sum_{j\in N} SC_j(N,v) \neq v(N)$, taking $\alpha = \frac{v(N)}{\sum_{j\in N} SC_j(N,v) - v(N)}$ yields

$$\begin{aligned}
&\alpha \sum_{i\in N\backslash\{h\}} \left(PANSC_i(N,v) - PANSC_i(N\backslash\{h\}, v^{-h})\right) \\
=\ & \alpha\left(v(N) - PANSC_h(N,v) - v(N\backslash\{h\})\right) \\
=\ & \alpha\left(SC_h(N,v) - PANSC_h(N,v)\right) \\
=\ & \alpha\left(SC_h(N,v) - \frac{SC_h(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N)\right) \\
=\ & \alpha \cdot SC_h(N,v) \cdot \left(1 - \frac{v(N)}{\sum_{j\in N} SC_j(N,v)}\right) \\
=\ & \alpha \cdot SC_h(N,v) \cdot \left(\frac{\sum_{j\in N} SC_j(N,v) - v(N)}{\sum_{j\in N} SC_j(N,v)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{v(N)}{\sum_{j\in N} SC_j(N,v) - v(N)}\right) \cdot SC_h(N,v) \cdot \left(\frac{\sum_{j\in N} SC_j(N,v) - v(N)}{\sum_{j\in N} SC_j(N,v)}\right) \\
&= \frac{SC_h(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) \\
&= PANSC_h(N,v),
\end{aligned}
$$

where in the first equality we twice use that the PANSC value is efficient (once on game (N,v) and once on game $(N\backslash\{h\}, v^{-h})$). This shows that the PANSC value satisfies weak balanced externalities if $\sum_{j\in N} SC_j(N,v) \neq v(N)$.

Case (ii). If $\sum_{j\in N} SC_j(N,v) = v(N)$, then $PANSC_i(N,v) = SC_i(N,v)$ for all $i \in N$, and thus we have by efficiency of the PANSC value that

$$
\begin{aligned}
&\sum_{i\in N\backslash\{h\}} \left(PANSC_i(N,v) - PANSC_i(N\backslash\{h\}, v^{-h})\right) \\
&= v(N) - PANSC_h(N,v) - v(N\backslash\{h\}) \\
&= SC_h(N,v) - SC_h(N,v) \\
&= 0,
\end{aligned}
$$

implying that weak balanced externalities does not have any bite.

To prove the 'only if' part, suppose that a value ψ satisfies efficiency and weak balanced externalities on $\mathcal{G}^{\geq 2}_{sc+}$. Let $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$.

Case (i). If $\sum_{i\in N\backslash\{h\}}(\psi_i(N,v) - \psi_i(N\backslash\{h\}, v^{-h})) \neq 0$, then efficiency and weak balanced externalities together imply that there is $\alpha \in \mathbb{R}$ such that for any $h \in N$,

$$
\begin{aligned}
\psi_h(N,v) &= \alpha \sum_{i\in N\backslash\{h\}} (\psi_i(N,v) - \psi_i(N\backslash\{h\}, v^{-h})) \\
&= \alpha \left(v(N) - \psi_h(N,v) - v(N\backslash\{h\})\right),
\end{aligned}
$$

and thus

$$(1+\alpha)\psi_h(N,v) = \alpha \left(v(N) - v(N\backslash\{h\})\right),$$

meaning

$$\psi_h(N,v) = \frac{\alpha}{1+\alpha} \left(v(N) - v(N\backslash\{h\})\right).$$

Efficiency determines that

$$\sum_{h\in N} \psi_h(N,v) = \sum_{h\in N} \frac{\alpha}{1+\alpha} \left(v(N) - v(N\backslash\{h\})\right) = \frac{\alpha}{1+\alpha} \sum_{h\in N} SC_h(N,v) = v(N),$$

implying that $\alpha = \frac{v(N)}{\sum_{h\in N} SC_h(N,v) - v(N)}$, and thus

$$\psi_h(N,v) = PANSC_h(N,v) \quad \text{for all } h \in N.$$

Case (ii). If $\sum_{i\in N\backslash\{h\}}(\psi_i(N,v) - \psi_i(N\backslash\{h\}, v^{-h})) = 0$, then by efficiency, we have $v(N) - \psi_h(N,v) - v(N\backslash\{h\}) = 0$, implying that $\psi_h(N,v) = v(N) - v(N\backslash\{h\}) =$

$SC_h(N,v) = PANSC_h(N,v)$. □

Proof of Proposition 3.3. For every $(N,v) \in \mathcal{G}^N_{nz+}$ and $i \in N$,

$$\begin{aligned} PANSC_i(N,v^*) &= \frac{v^*(N) - v^*(N\backslash\{i\})}{\sum_{j\in N}(v^*(N) - v^*(N\backslash\{j\}))}v^*(N) \\ &= \frac{v(N) - (v(N) - v(\{i\}))}{\sum_{j\in N}(v(N) - (v(N) - v(\{j\})))}v(N) \\ &= \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}v(N) \\ &= PD_i(N,v). \end{aligned}$$

□

Proof of Theorem 3.4. It is obvious that ψ^β is efficient. To show that this solution satisfies mollified weak balanced externalities, let $(N,v),(N,v'_\beta) \in \mathcal{G}_{\beta+}$ and $\beta \in \mathbb{R}$ be such that $v'_\beta(S) = \beta v(S) + (1-\beta)v^*(S)$ for all $S \subseteq N$. We consider two cases.

Case (i). If $\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\})) \neq v(N)$, taking

$$\alpha = \frac{v(N)}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\})) - v(N)}$$

for any $h \in N$, we have

$$\begin{aligned} &\alpha\left(\sum_{i\in N\backslash\{h\}} \psi^\beta_i(N,v) - v'_\beta(N\backslash\{h\})\right) \\ = &\ \alpha\big(v(N) - \psi^\beta_h(N,v) - \beta v(N\backslash\{h\}) - (1-\beta)v^*(N\backslash\{h\})\big) \\ = &\ \alpha\big(v(N) - \psi^\beta_h(N,v) - \beta v(N\backslash\{h\}) - (1-\beta)v(N) + (1-\beta)v(\{h\})\big) \\ = &\ \alpha\big(\beta(v(N) - v(N\backslash\{h\})) + (1-\beta)v(\{h\}) - \psi^\beta_h(N,v)\big) \\ = &\ \alpha\big(\beta v^*(\{h\}) + (1-\beta)v(\{h\}) - \psi^\beta_h(N,v)\big) \\ = &\ \alpha\left(\beta v^*(\{h\}) + (1-\beta)v(\{h\}) - \frac{\beta v^*(\{h\}) + (1-\beta)v(\{h\})}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}v(N)\right) \\ = &\ \alpha\left(\beta v^*(\{h\}) + (1-\beta)v(\{h\})\right)\left(1 - \frac{v(N)}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}\right) \\ = &\ \alpha\left(\beta v^*(\{h\}) + (1-\beta)v(\{h\})\right)\left(\frac{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\})) - v(N)}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}\right) \\ = &\ \frac{\beta v^*(\{h\}) + (1-\beta)v(\{h\})}{\sum_{j\in N}\left(\beta v^*(\{j\}) + (1-\beta)v(\{j\})\right)}v(N) \\ = &\ \psi^\beta_h(N,v), \end{aligned}$$

where the first equality holds from efficiency of ψ^β, and the eighth equality follows from substituting α.

Case (ii). If $\sum_{j\in N}(\beta v^*(\{j\})+(1-\beta)v(\{j\}))=v(N)$, then $\psi_i^\beta(N,v)=\beta v^*(\{i\})+(1-\beta)v(\{i\})$, and thus $\sum_{i\in N\setminus\{h\}}\psi_i^\beta(N,v)-v'_\beta(N\setminus\{h\},v^{-h}) = v(N)-\psi_h^\beta(N,v)-(\beta v(N\setminus\{h\})+(1-\beta)v^*(N\setminus\{h\}))=v(N)-(\beta v^*(\{h\})+(1-\beta)v(\{h\}))-\beta(v(N)-v^*(\{h\}))-(1-\beta)(v(N)-v(\{h\}))=0$, implying that mollified weak balanced externalities does not have any bite. Altogether, ψ^β satisfies mollified weak balanced externalities.

To prove the 'only if' part, let ψ be a solution satisfying efficiency and mollified weak balanced externalities on $\mathcal{G}_{\beta+}$. Let $(N,v)\in\mathcal{G}_{\beta+}$. We consider two cases.

Case (i). If $\sum_{i\in N\setminus\{h\}}\psi_i(N,v)-v'_\beta(N\setminus\{h\})\neq 0$, then efficiency and mollified weak balanced externalities imply that there is $\alpha\in\mathbb{R}$ such that for $h\in N$,

$$\begin{aligned}\psi_h(N,v) &= \alpha\Big(\sum_{i\in N\setminus\{h\}}\psi_i(N,v)-v'_\beta(N\setminus\{h\})\Big)\\ &= \alpha\big(\beta v^*(\{h\})+(1-\beta)v(\{h\})-\psi_h(N,v)\big),\end{aligned}$$

where the second equality holds from the first four equalities shown in the existence part. Thus,

$$\psi_h(N,v)=\frac{\alpha}{1+\alpha}\big(\beta v^*(\{h\})+(1-\beta)v(\{h\})\big).$$

Efficiency determines that

$$\sum_{h\in N}\psi_h(N,v)=\frac{\alpha}{1+\alpha}\sum_{h\in N}(\beta v^*(\{h\})+(1-\beta)v(\{h\}))=v(N),$$

and thus

$$\frac{\alpha}{1+\alpha}=\frac{v(N)}{\sum_{h\in N}(\beta v^*(\{h\})+(1-\beta)v(\{h\}))},$$

which yields the desired formula.

Case (ii). If $\sum_{i\in N\setminus\{h\}}\psi_i(N,v)-v'_\beta(N\setminus\{h\})=0$, then by efficiency, $v(N)-\psi_h(N,v)-v'_\beta(N\setminus\{h\})=0$, implying that $\psi_h(N,v)=v(N)-v'_\beta(N\setminus\{h\})=v(N)-\big(\beta v(N\setminus\{h\})+(1-\beta)(v(N)-v(\{h\}))\big)=\beta(v(N)-v(N\setminus\{h\}))+(1-\beta)v(\{h\})=\beta v^*(\{h\})+(1-\beta)v(\{h\})$. By efficiency, $\sum_{h\in N}\psi_h(N,v)=\sum_{h\in N}[\beta v^*(\{h\})+(1-\beta)v(\{h\})]=v(N)>0$ since $(N,v)\in\mathcal{G}_{\beta+}$. Therefore,

$$\begin{aligned}\psi_i(N,v) &= \beta v^*(\{i\})+(1-\beta)v(\{i\})\\ &= \frac{\beta v^*(\{i\})+(1-\beta)v(\{i\})}{v(N)}v(N)\\ &= \frac{\beta v^*(\{i\})+(1-\beta)v(\{i\})}{\sum_{k\in N}(\beta v^*(\{k\})+(1-\beta)v(\{k\}))}v(N),\end{aligned}$$

as desired. □

Proof of Proposition 3.4. Let $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$ and $j \in N$. First, we remark that the class of games $\mathcal{G}^{\geq 2}_{sc+}$ is subgame closed under the complement reduced game operator, namely, if $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$ for $|N| \geq 3$, then $(N\backslash\{j\}, v^x) \in \mathcal{G}^{\geq 2}_{sc+}$ for any $j \in N$ and $x = PANSC(N,v)$, because $SC_i(N\backslash\{j\}, v^x) = v^x(N\backslash\{j\}) - v^x(N\backslash\{i,j\}) = v(N) - PANSC_j(N,v) - (v(N\backslash\{i\}) - PANSC_j(N,v)) = v(N) - v(N\backslash\{i\}) = SC_i(N,v)$ for any $i \in N\backslash\{j\}$.[7] Next, we have, for any $i \in N\backslash\{j\}$,

$$\begin{aligned}
PANSC_i(N\backslash\{j\}, v^x) &= \frac{SC_i(N\backslash\{j\}, v^x)}{\sum_{k\in N\backslash\{j\}} SC_k(N\backslash\{j\}, v^x)} v^x(N\backslash\{j\}) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} (v(N) - x_j) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} \left(v(N) - \frac{SC_j(N,v)}{\sum_{k\in N} SC_k(N,v)} v(N)\right) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} \left(1 - \frac{SC_j(N,v)}{\sum_{k\in N} SC_k(N,v)}\right) v(N) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} \cdot \frac{\sum_{k\in N\backslash\{j\}} SC_k(N,v)}{\sum_{k\in N} SC_k(N,v)} v(N) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N} SC_k(N,v)} v(N) \\
&= PANSC_i(N,v).
\end{aligned}$$

□

Proof of Theorem 3.5. It is straightforward that the PANSC value satisfies dual proportional standardness. Complement consistency follows from Proposition 3.4.

To prove the 'only if' part, let ψ be a value on $\mathcal{G}^{\geq 2}_{sc+}$ which satisfies the two axioms. Let $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$, $x = PANSC(N,v)$, and $y = \psi(N,v)$. We will show that $x = y$. If $|N| = 2$, $x = y$ follows from dual proportional standardness. Suppose, by induction, that $PANSC_i(N',v) = \psi_i(N',v)$ holds for any game (N',v) with $|N'| < |N|$.

Take any $i \in N$ and $j \in N\backslash\{i\}$, and consider (N,v) and the complement reduced games $(N\backslash\{j\}, v^x)$, $(N\backslash\{j\}, v^y)$. We have

$$\begin{aligned}
x_i - y_i &= PANSC_i(N,v) - \psi_i(N,v) \\
&= PANSC_i(N\backslash\{j\}, v^x) - \psi_i(N\backslash\{j\}, v^y) \\
&= PANSC_i(N\backslash\{j\}, v^x) - PANSC_i(N\backslash\{j\}, v^y) \\
&= \frac{SC_i(N\backslash\{j\}, v^x)}{\sum_{k\in N\backslash\{j\}} SC_k(N\backslash\{j\}, v^x)} v^x(N\backslash\{j\}) - \frac{SC_i(N\backslash\{j\}, v^y)}{\sum_{k\in N\backslash\{j\}} SC_k(N\backslash\{j\}, v^y)} v^y(N\backslash\{j\}) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} v^x(N\backslash\{j\}) - \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} v^y(N\backslash\{j\}) \\
&= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} (v(N) - x_j) - \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} (v(N) - y_j)
\end{aligned}$$

[7]This equality is also used in the proof of Theorem 3.5. Notice that the class of games $\mathcal{G}^{\geq 2}_{sc+}$ is subgame closed under this reduced game operator for any value ψ.

$$= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)}(-x_j+y_j), \tag{3.5}$$

where the second equality follows from the PANSC value and ψ satisfying complement consistency, the third equality follows from the induction hypothesis, and the fifth equality follows similar as in the proof of Proposition 3.4, referring to Footnote 7.

Summing up (3.5) over all $i \in N\backslash\{j\}$ yields

$$\sum_{i\in N\backslash\{j\}}(x_i-y_i) = \sum_{i\in N\backslash\{j\}} \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)}(-x_j+y_j) = -x_j+y_j,$$

and thus

$$\sum_{j\in N\backslash\{i\}}(-x_j+y_j) = x_i - y_i. \tag{3.6}$$

Summing up (3.5) over all $j \in N\backslash\{i\}$ yields

$$\begin{aligned}\sum_{j\in N\backslash\{i\}}(x_i-y_i) &= (|N|-1)(x_i-y_i)\\ &= \frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)}\sum_{j\in N\backslash\{i\}}(-x_j+y_j).\end{aligned} \tag{3.7}$$

Together (3.6) and (3.7) imply that

$$\left(|N|-1-\frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)}\right)(x_i-y_i) = 0.$$

Since $SC_k(N,v) > 0$ for any $k \in N$ and $i \in N\backslash\{j\}$, we have

$$|N|-1-\frac{SC_i(N,v)}{\sum_{k\in N\backslash\{j\}} SC_k(N,v)} \neq 0 \text{ for any } i \in N \text{ and } j \in N\backslash\{i\}.$$

Therefore, we have $x_i = y_i$. □

Proof of Lemma 3.1. It is clear that the 'if' part is satisfied. To show the 'only if' part, suppose that ψ is a value satisfying dual proportionality, the inessential game property for two-player games, and continuity for two-player games. Let (N,v) be an arbitrary game in the class $\mathcal{G}^2_{sc+}$ with $N = \{i,j\}$.

If $v(N) \neq 0$, let (N,v') be an additive game such that $SC_i(N,v') = \alpha SC_i(N,v)$, $SC_j(N,v') = \alpha SC_j(N,v)$ and $v'(N) = v(N)$. Clearly, $\alpha = \frac{v(N)}{SC_i(N,v)+SC_j(N,v)} \neq 0$. Dual proportionality and the inessential game property for two-player games imply that

$$\psi_k(N,v) = \psi_k(N,v') = \alpha SC_k(N,v) = \frac{SC_k(N,v)}{SC_i(N,v)+SC_j(N,v)}v(N) \text{ for all } k \in \{i,j\}.$$

If $v(N)=0$, then continuity for two-player games implies that

$$\psi(N,v)=PANSC(N,v)=\mathbf{0}.$$

$\square$

Proof of Proposition 3.5 . The 'if' part is straightforward. To prove the 'only if' part, suppose that ψ is a value on $\mathcal{G}^2_{sc\mathbb{Q}}$ that satisfies the two axioms. Let $(N,v)\in\mathcal{G}^2_{sc\mathbb{Q}}$ with $N=\{i,j\}$. For any $\alpha\in\mathbb{Q}$, consider the game (N,v^α) defined by $v^\alpha(\{i\})=v(\{i\})+(\alpha-1)v(N)$, $v^\alpha(\{j\})=v(\{j\})+(\alpha-1)v(N)$ and $v^\alpha(N)=\alpha v(N)$. Clearly, $(N,v^\alpha)\in\mathcal{G}^2_{sc\mathbb{Q}}$.

On the one hand, we show that $\psi(N,v^\alpha)=\alpha\psi(N,v)$ for all $\alpha\in\mathbb{Z}$. For games (N,v) and (N,v^0), using relevant additivity we obtain $\psi(N,v)+\psi(N,v^0)=\psi(N,v+I_{v^0})=\psi(N,v)$, which implies $\psi(N,v^0)=\mathbf{0}$. For any $k\in\mathbb{Z}^+$, since $(N,v^k)=(N,v^{k-1}+I_v)$, relevant additivity implies $\psi(N,v^k)=\psi(N,v^{k-1})+\psi(N,v)$. This recursion formula yields $\psi(N,v^k)=k\psi(N,v)$. For any $k\in\mathbb{Z}^-$, since $(N,v^k+I_v)=(N,v^{k+1})$, relevant additivity then implies $\psi(N,v^k)+\psi(N,v)=\psi(N,v^{k+1})$. This recursion formula yields $\psi(N,v^k)=-|k|\psi(N,v)=k\psi(N,v)$. Thus, we conclude that $\psi(N,v^k)=k\psi(N,v)$ for all $k\in\mathbb{Z}$.

On the other hand, we have that $\psi(N,v)=m\psi(N,v^{\frac{1}{m}})$ for all $m\in\mathbb{Z}\backslash\{0\}$. Indeed, for any $m\in\mathbb{Z}^+$, since $(N,v^{\frac{m-k}{m}})=(N,v^{\frac{m-k-1}{m}}+I_{v^{\frac{1}{m}}})$ for all $k\in\{0,1,\ldots,m-1\}$, using relevant additivity and then summing these equations yield $\psi(N,v)=m\psi(N,v^{\frac{1}{m}})$. Similarly, for any $m\in\mathbb{Z}^-$, since $(N,v^{\frac{m+k}{m}}+I_{v^{\frac{1}{m}}})=(N,v^{\frac{m+k+1}{m}})$ for all $k\in\{0,1,\ldots,|m|-1\}$, we have $\psi(N,v)=-|m|\psi(N,v^{\frac{1}{m}})+\psi(N,v^0)=m\psi(N,v^{\frac{1}{m}})$.

Since any rational number $\alpha\in\mathbb{Q}$ can be written as $\alpha=\frac{k}{m}$, where $k\in\mathbb{Z}$ and $m\in\mathbb{Z}\backslash\{0\}$. Therefore, $\psi(N,v^\alpha)=\psi(N,v^{\frac{k}{m}})=k\psi(N,v^{\frac{1}{m}})=\frac{k}{m}\psi(N,v)=\alpha\psi(N,v)$.

To show $\psi=PANSC$, we distinguish two cases: $v(N)=0$ and $v(N)\neq0$. For every game (N,v) with $v(N)=0$, $\psi(N,v)=\psi(N,v^0)=\mathbf{0}=PANSC(N,v)$. For every game (N,v) with $v(N)\neq0$, take $\alpha=\frac{v(N)-v(\{i\})+v(N)-v(\{j\})}{v(N)}$, and then (N,v^α) is an additive game. The inessential game property for two-player games implies that $\psi_k(N,v^\alpha)=v^\alpha(\{k\})=v(\{k\})+(\alpha-1)v(N)$ for all $k\in\{i,j\}$. Hence, $\psi_i(N,v)=\frac{1}{\alpha}\psi_i(N,v^\alpha)=\frac{v(\{i\})-v(N)}{\alpha}+v(N)=\frac{v(N)-v(\{j\})}{v(N)-v(\{i\})+v(N)-v(\{j\})}v(N)=PANSC_i(N,v)$ and $\psi_j(N,v)=\frac{v(N)-v(\{i\})}{v(N)-v(\{i\})+v(N)-v(\{j\})}v(N)=PANSC_j(N,v)$, as desired. $\square$

Proof of Proposition 3.6. For any $(N,v)\in\mathcal{G}^{\geq2}_{sc+}$ and $i\in N$,

$$PANSC_i(N,v)=SC_i(N,v)+\frac{SC_i(N,v)}{\sum_{k\in N}SC_k(N,v)}[v(N)-\sum_{k\in N}SC_k(N,v)].$$

Comparing this equation with (1.2), we have that $EANSC(N, v) = PANSC(N, v)$ if and only if

$$[v(N) - \sum_{k \in N} SC_k(N, v)][\frac{SC_i(N, v)}{\sum_{k \in N} SC_k(N, v)} - \frac{1}{n}] = 0 \quad \text{for all } i \in N,$$

and thus $v(N) = \sum_{k \in N} SC_k(N, v)$ or $v(N \backslash \{i\}) = v(N \backslash \{j\})$ for all $i, j \in N$. □

Proof of Theorem 3.6. It is straightforward to show that the EANSC value satisfies complement consistency and standardness. To show the 'only if' part, suppose that ψ is a value satisfying complement consistency and standardness.

If $|N| = 2$, then $\psi(N, v) = EANSC(N, v)$ follows from standardness.

Proceeding by induction, for $|N| \geq 3$, suppose that $\psi(N', w) = EANSC(N', w)$ whenever $|N'| = |N| - 1$. Take any $i, j \in N$ such that $i \neq j$. Let $x = \psi(N, v)$ and $y = EANSC(N, v)$. For the two reduced games $(N \backslash \{j\}, v^x)$ and $(N \backslash \{j\}, v^y)$, by the induction hypothesis, we have

$$\begin{aligned} x_i - y_i &= \psi_i(N \backslash \{j\}, v^x) - EANSC_i(N \backslash \{j\}, v^y) \\ &= EANSC_i(N \backslash \{j\}, v^x) - EANSC_i(N \backslash \{j\}, v^y). \end{aligned} \tag{3.8}$$

By definition of the EANSC value and the complement reduced game, we have

$$\begin{aligned} & EANSC_i(N \backslash \{j\}, v^x) - EANSC_i(N \backslash \{j\}, v^y) \\ = \; & SC_i(N \backslash \{j\}, v^x) + \frac{1}{|N| - 1}[v^x(N \backslash \{j\}) - \sum_{k \in N \backslash \{i\}} SC_k(N \backslash \{j\}, v^x)] \\ & - SC_i(N \backslash \{j\}, v^y) - \frac{1}{|N| - 1}[v^y(N \backslash \{j\}) - \sum_{k \in N \backslash \{i\}} SC_k(N \backslash \{j\}, v^y)] \\ = \; & v(N) - v(N \backslash \{i\}) + \frac{1}{|N| - 1}[v(N) - x_j - \sum_{k \in N \backslash \{i\}} (v(N) - v(N \backslash \{k\})] \\ & - (v(N) - v(N \backslash \{i\})) - \frac{1}{|N| - 1}[v(N) - y_j - \sum_{k \in N \backslash \{i\}} (v(N) - v(N \backslash \{k\})] \\ = \; & \frac{1}{|N| - 1}(y_j - x_j). \end{aligned}$$

Together with (3.8), this implies that, for all $i, j \in N$ with $i \neq j$,

$$x_i - y_i = \frac{1}{|N| - 1}(y_j - x_j). \tag{3.9}$$

Summing (3.9) over all $i \in N \backslash \{j\}$ yields $\sum_{i \in N \backslash \{j\}} (x_i - y_i) = y_j - x_j$, which implies

$$\sum_{i \in N} (x_i - y_i) = 0. \tag{3.10}$$

On the other hand, (3.9) can be written as $(|N|-1)(x_i - y_i) = y_j - x_j$. Summing this equality over all $j \in N\backslash\{i\}$, we have

$$(|N|-1)^2(x_i - y_i) = \sum_{j\in N\backslash\{i\}} (y_j - x_j). \tag{3.11}$$

Together with (3.10) and (3.11), it holds that $|N|(|N|-2)(x_i - y_i) = 0$. Thus, $x_i - y_i = 0$ for all $i \in N$. This shows that $\psi(N,v) = EANSC(N,v)$. □

Proof of Proposition 3.7. Consider cost allocation problem (N,c,b).

(i) Suppose that $b(j) = 0$ for all $j \in N$. Then, the SCRB solution becomes, for all $i \in N$,

$$\begin{aligned} SCRB_i(N,c,b) &= SC_i(N,c) + \frac{-SC_i(N,c)\cdot NSC(N,c)}{\sum_{j\in N}(-SC_j(N,c))} \\ &= \frac{SC_i(N,c)}{\sum_{j\in N} SC_j(N,c)} \cdot c(N). \end{aligned}$$

For the associated anti-game (N,v), since $SC_j(N,v) = -SC_j(N,c)$ for all $j \in N$, and $c(N) = -v(N)$, we have

$$\begin{aligned} SCRB_i(N,c,b) &= -\frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) \\ &= -PANSC_i(N,v), \end{aligned}$$

and thus the cost allocation determined by the SCRB method coincides with (the negative of) our PANSC value applied to the associated anti-game.

(ii) Suppose that $b(j) \geq c(\{j\})$ for all $j \in N$. In that case, for all $i \in N$,

$$\begin{aligned} SCRB_i(N,c,b) &= ACA_i(N,c,b) \\ &= SC_i(N,c) + \frac{(c(\{i\}) - SC_i(N,c))\cdot NSC(N,c)}{\sum_{j\in N}(c(\{j\}) - SC_j(N,c))}. \end{aligned}$$

For the associated cost saving game (N,v), since $SC_j(N,v) = v(N) - v(N\backslash\{j\}) = \sum_{k\in N} c(\{k\}) - c(N) - \sum_{k\in N\backslash\{j\}} c(\{k\}) + c(N\backslash\{j\}) = c(\{j\}) - SC_j(N,c)$ for all $j \in N$, $c(N) = \sum_{k\in N} c(\{k\}) - v(N)$, and thus

$$\begin{aligned} NSC(N,c) &= c(N) - \sum_{k\in N} SC_k(N,c) \\ &= \sum_{k\in N} c(\{k\}) - v(N) - \sum_{k\in N}(c(\{k\}) - SC_k(N,v)) \\ &= \sum_{k\in N} SC_k(N,v) - v(N) \\ &= -NSC(N,v), \end{aligned}$$

we have

$$
\begin{aligned}
c(\{i\}) - SCRB_i(N,c,b) &= c(\{i\}) - SC_i(N,c) - \frac{(c(\{i\}) - SC_i(N,c)) \cdot NSC(N,c)}{\sum_{j\in N}(c(\{j\}) - SC_j(N,c))} \\
&= SC_i(N,v) + \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} NSC(N,v) \\
&= \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) \\
&= PANSC_i(N,v).
\end{aligned}
$$

and thus the cost allocation determined by the SCRB method coincides with the individual cost of each player minus the cost share allocated by our PANSC value applied to the associated saving game. □

3.9 Conclusion

One of the most popular values in *queueing problems* is the *minimal transfer rule*, which is obtained by applying the Shapley value to an associated TU-game. Since queueing games are so-called *2-games*, this minimal transfer rule coincides also with other TU-game values, such as the pre-nucleolus and the τ-value of the associated queueing game. In van den Brink and Chun (2012), the minimal transfer rule is characterized by *efficiency*, *Pareto indifference* and *balanced cost reduction*. The last axiom requires that the payoff of any player is equal to the total externality she inflicts on the other players with its presence, i.e. it equals the sum of all changes in the payoffs of all other players if she leaves the queueing problem.

In this chapter, we have evaluated which value can be obtained if we extend the balanced cost reduction property from queueing problems to TU-games. After extending the characterization result to the class of *2-games*, we show that extending this axiom in a straightfoward way to general TU-games, is incompatible with efficiency. Keeping as close as possible to the idea behind balanced cost reduction, we weaken the axiom by requiring that every player's payoff is the same *fraction* of its total externality inflicted on the other players. This weakening, which we call *weak balanced externalities*, turns out to be compatible with efficiency. More specifically, the unique efficient value that satisfies this weaker property is the *PANSC value*, which allocates the payoffs proportional to the *separable cost* of the players. Since the PANSC value is the dual of the PD value, characterizations of the PANSC value can be derived from that of the PD value, see, for example, Section 3.6. We also have characterized a class of values that has the PANSC value and the PD value as polar cases, as in Theorem 3.4. This value allocates the worth of the grand coalition among the players in proportion to their affine combination of the stand-alone worth and the separable cost.

Many interesting topics are wide open for future studies. We only list a few in the below. (i) Apply the PANSC and PD values to the almost diminishing marginal contributions games (Leng et al., 2021) with positive stand-alone worths. (ii) Study characterizations of the convex combinations of the PANSC value and the PD value. (iii) Study the PANSC and EANSC values of the broadcasting games (Bergantiños and Moreno-Ternero, 2020a; Bergantiños and Moreno-Ternero, 2020b; Bergantiños and Moreno-Ternero, 2021) based on the theory of 2-games (or queueing problems).

Another future research goes to cooperative strategic games in Kohlberg and Neyman (2020), as an extension of two-person complete-information strategic games introduced by Kalai and Kalai (2013). In Kohlberg and Neyman (2020), they present two highly simplified game models of a public official who has the authority to make decisions in matters of financial importance to private individuals or companies. One of them, called "authority to issue licenses", is described as follows. Each player $i \in N$ seeks approval for a project; another player A has the authority to approve up to k projects ($1 \leq k \leq |N|$). Kohlberg and Neyman (2020) consider it as a strategic game where player A can choose any subset of players of size at most k, while every player $i \in N$ has no strategic choices; the payoff of player A is zero, while the payoff of player $i \in N$ is α_i or 0 (assume that $\alpha_1 \geq \ldots \geq \alpha_n \geq 0$), depending whether her project is approved or not. The value for such games can be useful in designing systems of incentives and penalties intended to deter bribery. As stated by Kohlberg and Neyman (2020), cooperative strategic games are quite different from cooperative games. Clearly, this game can also be modeled as an extension of 2-game, which only coalitions of size two and more than $k+1$ can have a nonzero dividend. Thus, it might be possible to investigate this kind of cooperative strategic games based on the EANSC value, the PANSC value and the Shapley value of the corresponding cooperative games.

Chapter 4

Compromising between the Proportional and Equal Division Values

4.1 Introduction

The central question in TU-games is how to allocate the worth of the grand coalition over the players. Undoubtedly, an extreme egalitarian value is the *equal division (ED) value* (characterized in van den Brink (2007)), which allocates the total worth equally among all players. In front of this equality principle, the *proportional division (PD) value* relies on the proportionality principle: it gives payoffs in proportion to players' stand-alone worth, see Chapter 2.

Proportionality and equality are fundamental principles in various allocation problems. Specifically, in bankruptcy problems where the available amount of a resource that is to be allocated is not enough to honor all players' claims on it, a pioneering work was done by O'Neill (1982) which shows that the proportional and equal rules are two prominent concepts both in practice and in theory. Subsequently, some research has focused on comparing and characterizing different ways of compromising between the proportional and equal division rules for bankruptcy problems and other related problems. The reader is referred to Moulin (1987), Giménez-Gómez and Peris (2014), and Thomson (2015b), and to Thomson (2003) and Thomson (2015a) for overviews of this literature. As shown in O'Neill (1982), a bankruptcy problem can be modeled as a TU-game. However, the sizable literature does not touch the issue of generalizing values that compromise between proportionality and equality principles for general TU-games.

In this chapter, which is based on Zou et al. (2020a), we introduce a family of values for TU-games that offers a simple yet flexible compromise between proportionality and equality principles. Our values, which we call *α-mollified values*, contain convex combinations of the ED and PD values[1] introduced in Moulin (1987) for

[1]The equal division value and the proportional division value are, respectively, called the equal sharing rule and the proportional sharing rule in Moulin (1987).

a special type of game. Specifically, the *α-mollified value* determines the payoff allocation in two steps. First, linear functions are defined that associate a real number (the initial share) to every TU-game. Second, the initial share is allocated over the players in proportion to their stand-alone worth, and the residual of the worth of the grand coalition is split equally over all players. This value reduces to the ED value or the PD value when the initial share is zero or the worth of the grand coalition, respectively. After introducing these values, we first provide an axiomatization of the family of α-mollified values, as well as for affine combinations of the ED value and the equal surplus division (ESD) value (also known as the CIS value in Driessen and Funaki (1991)).

Our second result identifies which members of our family satisfy projection consistency (see, e.g., Funaki and Yamato (2001), van den Brink and Funaki (2009), van den Brink et al. (2016), Calleja and Llerena (2017), and Calleja and Llerena (2019)). The consistency principle has been successfully applied to characterize a wide variety of value concepts for TU-games. Given a payoff vector for some initial game, and given a subgroup of players, a so-called *reduced game* among these players is constructed where the worth of a coalition of remaining players depends on what they can earn with leaving players, but also taking account of the payoffs assigned to the leaving players. A value is *consistent* if it selects the same payoff allocation for any reduced game. Different values satisfy different reduced game consistency properties, where the difference is with respect to the way the reduced game is defined. In the literature, various consistency properties are applied, using different reduced games, which together with some properties characterize a unique point-valued or set-valued solution. Some of the contributions on various solutions can be found in Hart and Mas-Colell (1989), van den Brink and Funaki (2009), van den Brink et al. (2016), Calleja and Llerena (2017), and Calleja and Llerena (2019), and the surveys of Driessen and Funaki (1991) and Thomson (2011a).[2]

Instead of characterizing a unique value among all values, in this paper we use consistency to identify a subclass of values from our class of α-mollified values. More precisely, we focus on projection consistency exerting on a generalization of the α-mollified values in which the initial share is measured as a general (possibly asymmetric) linear function with respect to the worths of all coalitions. As it turns out, projection consistency singles out either the PD value, or an egalitarian value (being affine combination of the ED and ESD values). That is, through the α-mollified values depend on all coalition worths, projection consistency just singles out these special cases. This result provides an advantage of the proportional division value and egalitarian values above other α-mollified (generalized) values. The proofs of these results are technical, but follow a novel analytical approach.

[2]In contrast to the remarkable research on characterizing values, there are relatively few, yet significant works on constructing consistent extensions of two-claimant rules, in the context of claims problems. Such works mainly focus on identifying which members of the two-claimant family can be generalized to general populations by requiring consistency; we refer to Thomson (2008), Thomson (2013), and Thomson (2015b).

Besides an axiomatic approach, we also provide a procedural implementation of the α-mollified values by designing a dynamic allocation process based on a one-by-one formation of the grand coalition, which is similar to that of the weighted ENSC value in Hou et al. (2019).

This chapter is organized as follows. Section 4.2 recalls definitions and notation. Section 4.3 introduces the concept of the α-mollified value. Section 4.4 identifies a characterization. In Section 4.5, we focus on projection consistency exerting on our family. Section 4.6 gives a procedural implementation. Section 4.7 shows the logical independence of the axioms in a characterization result. The proofs are provided in Section 4.8. Section 4.9 concludes.

4.2 Definitions and notation

We recall some definitions from Chapter 1 that are used in this chapter. Recall that $\mathcal{G}^N_{nz}$ denotes the class that consists of all individually positive and individually negative games on a specific player set N, i.e., $\mathcal{G}^N_{nz} = \{(N,v) \in \mathcal{G}^N \mid v(\{i\}) > 0 \text{ for all } i \in N, \text{ or } v(\{i\}) < 0 \text{ for all } i \in N\}$. $\mathcal{G}^{\geq 2}_{nz}$ denotes the class of games in $\mathcal{G}^N_{nz}$ with at least two players, i.e. $\mathcal{G}^{\geq 2}_{nz} = \{(N,v) \in \mathcal{G} \mid |N| \geq 2 \text{ and } [v(\{i\}) > 0 \text{ for all } i \in N, \text{ or } v(\{i\}) < 0 \text{ for all } i \in N]\}$. Besides, we introduce three additional notation. Let $\mathcal{G}^N_{nz+}$ and $\mathcal{G}^N_{nz-}$ denote the classes of individually positive and individually negative games on N, respectively. Let $\mathcal{G}^N_D$ denote the class $\{(N,v) \in \mathcal{G}^N_{nz} \mid v(\{i\}) \neq v(\{j\}) \text{ for some } i,j \in N\}$.

The *equal division (ED) value* on $\mathcal{C} \subseteq \mathcal{G}^{\geq 2}_{nz}$ is defined for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$ED_i(N,v) = \frac{1}{n}v(N).$$

The *equal surplus division (ESD) value* on $\mathcal{C} \subseteq \mathcal{G}^{\geq 2}_{nz}$ is defined for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$ESD_i(N,v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})].$$

The *proportional division (PD) value* on $\mathcal{C} \subseteq \mathcal{G}^{\geq 2}_{nz}$ is defined for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$PD_i(N,v) = \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} v(N).$$

For any real number $\beta \in [0,1]$, the convex combination of the ED value and the PD value with respect to β, introduced by Moulin (1987) for a special type of game, is given for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$\psi^{\beta}_i(N,v) = \beta\, PD_i(N,v) + (1-\beta) ED_i(N,v). \tag{4.1}$$

The following properties of values, stated in Chapter 1 for arbitrary subclasses of games, will be considered in this chapter.

- **Efficiency**. For all $(N,v) \in \mathcal{C} \subseteq \mathcal{G}_{nz}^{\geq 2}$, it holds that $\sum_{i\in N} \psi_i(N,v) = v(N)$.
- **Linearity**. For all $(N,v),(N,w) \in \mathcal{C} \subseteq \mathcal{G}_{nz}^{\geq 2}$ and $a,b \in \mathbb{R}$ such that $(N, av + bw) \in \mathcal{C}$, it holds that $\psi(N, av+bw) = a\psi(N,v) + b\psi(N,w)$.
- **Anonymity**. For all $(N,v) \in \mathcal{C} \subseteq \mathcal{G}_{nz}^{\geq 2}$, all permutation $\pi : N \to N$ and all $i \in N$ such that $(N,\pi v) \in \mathcal{C}$, it holds that $\psi_i(N,v) = \psi_{\pi(i)}(N,\pi v)$.
- **Continuity**. For all sequences of games $\{(N,w_k)\}$ of elements of $\mathcal{C} \subseteq \mathcal{G}_{nz}^{\geq 2}$ and every $(N,v) \in \mathcal{C}$ such that $\lim\limits_{k\to\infty}(N,w_k) = (N,v)$, it holds that $\lim\limits_{k\to\infty}\psi(N,w_k) = \psi(N,v)$.
- **Weak additivity**. For all $(N,v),(N,w) \in \mathcal{C} \subseteq \mathcal{G}_{nz}$ such that there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, if $(N, v+w) \in \mathcal{C}$ then $\psi(N,v+w) = \psi(N,v) + \psi(N,w)$.

4.3 The family of α-mollified values

We generalize Formula (4.1) for TU-games by defining a new value, called α-mollified value, that not only adopts the proportional and equal division principles, but also takes into account the worths of all coalitions.

Definition 4.1. Let $\alpha : \mathcal{C} \to \mathbb{R}$ be a linear and anonymous function on $\mathcal{C} \subseteq \mathcal{G}_{nz}$. That is, for any $(N,v) \in \mathcal{C}$, $\alpha(N,v) = \sum\limits_{S\subseteq N} \alpha_{|S|}^{|N|} v(S)$, where every parameter $\alpha_{|S|}^{|N|} \in \mathbb{R}$. The *$\alpha$-mollified value* on $\mathcal{C}$ is defined for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$\psi_i^\alpha(N,v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\alpha(N,v) + \frac{1}{n}\big(v(N) - \alpha(N,v)\big). \tag{4.2}$$

Dutta and Ray (1989) argue that all coalitions should be considered when formulating an (egalitarian) allocation in a TU-game. This is clearly not the case when one considers the equal or proportional division value, or any convex combination of them. However, since the linear function α used to define an α-mollified value depends on all coalition worths, the payoff allocation according to an α-mollified value might depend on all coalition worths. The idea of assigning numbers to TU-games, as done by the linear function α is similar to Hart and Mas-Colell (1989) who associate a real number with a TU-game, called *potential*, that is the expected normalized worth, being a linear function with respect to the worths of all coalitions. Casajus and Huettner (2014a) shows that the potential is the expected worth generated by some natural random partition of the player set.

An alternative formula for the α-mollified value is given by

$$\psi_i^\alpha(N,v) = \frac{v(N)}{n} + \frac{v(\{i\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})}{\frac{1}{n}\sum_{k\in N} v(\{k\})}\frac{\alpha(N,v)}{n}.$$

Then this value can be interpreted as assigning to every player an equal share in the worth of the grand coalition, corrected by a (positive or negative) amount that is based on the deviation of the stand-alone worth and the average stand-alone worth, and the number $\alpha(N, v)$.

Remark 4.1. Various functions $\alpha(N, v)$ give rise to various values. Some examples are the following:

(i) The ED value is obtained when $\alpha(N, v) = 0$.

(ii) The ESD value is obtained when $\alpha(N, v) = \sum_{k\in N} v(\{k\})$.

(iii) The family of affine combinations of the ESD value and ED value, i.e. $\psi = \beta ESD + (1-\beta)ED$, is obtained when $\alpha(N, v) = \beta \sum_{k\in N} v(\{k\})$, $\beta \in \mathbb{R}$.

(iv) The family of affine combinations of the PD value and ED value, i.e. $\psi = \beta PD + (1-\beta)ED$, is obtained when $\alpha(N, v) = \beta v(N)$, $\beta \in \mathbb{R}$.

(v) The family of affine combinations of the PD value and ESD value, i.e. $\psi = \beta PD + (1-\beta)ESD$, is obtained when $\alpha(N, v) = \beta v(N) + (1-\beta)\sum_{k\in N} v(\{k\})$, $\beta \in \mathbb{R}$.

(vi) The family of proportional surplus division values Zou et al. (2020b) given by

$$\psi_i^{\alpha}(N, v) = \frac{\beta}{n}\sum_{k\in N} v(\{k\}) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}[v(N) - \beta\sum_{k\in N} v(\{k\})]$$

is obtained when $\alpha(N, v) = v(N) - \beta\sum_{k\in N} v(\{k\})$, $\beta \in \mathbb{R}$.

If $\beta = 0$, this is the PD value, recently characterized by Zou et al. (2021); if $\beta = 1$, this is the egalitarian proportional surplus division value characterized by Zou et al. (2020b). Zou et al. (2020b) also characterize the families of values when $\beta \in \mathbb{R}$ and $\beta \in [0, 1]$.

(vii) When $\alpha(N, v) = v(N) - \frac{1}{2^n-1}\sum_{S\subseteq N} v(S)$, we have a new value given by

$$\psi_i^{\alpha}(N, v) = \frac{1}{n(2^n-1)}\sum_{S\subseteq N} v(S) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}[v(N) - \frac{1}{2^n-1}\sum_{S\subseteq N} v(S)],$$

which allocates the average coalition worth among all players equally and then allocates the remainder of the worth of the grand coalition in proportion to players' stand-alone worth.

4.4 Axiomatization of the family of α-mollified values

There are several approaches to justify values for TU-games. Two of these approaches are axiomatization and providing a dynamic process. In this section, we provide an axiomatization for the family of α-mollified values, as well as for affine combinations of the ESD and ED values.

In order to characterize the family of α-mollified values, we consider the following axioms. First, the balanced individual excess ratio property states that the ratio of the difference of the payoffs of any two players over the difference of their stand-alone worths is equal for any pair of players. To avoid dividing by zero, we formulate this axiom as follows.

- **Balanced individual excess ratio property**. For any $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$ and any $i,j,k \in N$, it holds that

$$(\psi_i(N,v) - \psi_j(N,v))(v(\{i\}) - v(\{k\})) = (\psi_i(N,v) - \psi_k(N,v))(v(\{i\}) - v(\{j\})). \tag{4.3}$$

For any game $(N,v) \in \mathcal{G}_D^N$, and players $i,j,h \in N$ with $v(\{i\}) \neq v(\{j\})$ and $v(\{i\}) \neq v(\{k\})$, (4.3) can be written as

$$\frac{\psi_i(N,v) - \psi_j(N,v)}{v(\{i\}) - v(\{j\})} = \frac{\psi_i(N,v) - \psi_k(N,v)}{v(\{i\}) - v(\{k\})},$$

where the denominators are nonzero for all $(N,v) \in \mathcal{G}_D^N$.

Under efficiency and continuity, the balanced individual excess ratio property characterizes the following family of values.

Proposition 4.1. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, the balanced individual excess ratio property, and continuity if and only if there exists a continuous function $g : \mathcal{G}_{nz}^N \to \mathbb{R}$ such that*

$$\psi_i(N,v) = \frac{v(N)}{n} + g(N,v)[v(\{i\}) - \frac{1}{n}\sum_{k \in N} v(\{k\})] \tag{4.4}$$

for all $(N,v) \in \mathcal{G}_{nz}^N$ and $i \in N$.

The proof of Proposition 4.1 and of all other results can be found in Section 4.8.

Remark 4.2. Proposition 4.1 still holds if the domain $\mathcal{G}_{nz}^N$ is replaced by $\mathcal{G}_{nz+}^N$ or $\mathcal{G}_{nz-}^N$, which can be obtained from the proof of this proposition.

Among the values characterized in Proposition 4.1, only affine combinations of the ESD value and the ED value satisfy the linearity axiom. As shown below, continuity is superfluous in this characterization result.

Theorem 4.1. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, the balanced individual excess ratio property, and linearity if and only if there is $\beta \in \mathbb{R}$ such that*

$$\psi_i(N,v) = \beta v(\{i\}) + \frac{1}{n}[v(N) - \sum_{k \in N} \beta v(\{k\})] \tag{4.5}$$

for all $(N,v) \in \mathcal{G}_{nz}^N$ and $i \in N$.

To characterize the family of α-mollified values on $\mathcal{G}^N_{nz+}$ or $\mathcal{G}^N_{nz-}$, we add the following axiom.

- **No advantageous reallocation across individuals**. For all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ and $T \subseteq N$ such that $v(S) = w(S)$ for all $S \subseteq N$ with $|S| \geq 2$, $v(\{i\}) = w(\{i\})$ for all $i \in N \backslash T$ and $\sum_{i\in T} v(\{i\}) = \sum_{i\in T} w(\{i\})$, it holds that $\sum_{i\in T} \psi_i(N,v) = \sum_{i\in T} \psi_i(N,w)$.

No advantageous reallocation across individuals (Moulin, 1987) states that no group of players benefits if reallocating their stand-alone worths among themselves is allowed. All efficient linear and symmetric values satisfy this axiom. Making use of a similar axiom, Ertemel and Kumar (2018) characterize an extension of the proportional rule for rationing problems, which is similar to our α-mollified value in that context.

This axiom characterizes the family of α-mollified values together with weak additivity, anonymity, and the axioms in Proposition 4.1.

Theorem 4.2. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N_{nz+}$ (respectively $\mathcal{G}^N_{nz-}$) satisfies efficiency, the balanced individual excess ratio property, continuity, weak additivity, anonymity, and no advantageous reallocation across individuals if and only if ψ belongs to the family of α-mollified values.*

Remark 4.3. If weak additivity is replaced by weak linearity, then the coefficient $\alpha^{|N|}_{|S|}$ for each $S \subseteq N$ in (4.2) must be the same on $\mathcal{G}^N_{nz+}$ and $\mathcal{G}^N_{nz-}$, and thus in this case the domain in Theorem 4.2 can be extended to the class $\mathcal{G}^N_{nz}$. Together with Theorem 4.1, we conclude that the affine combinations of the ESD value and the ED value are the only linear values in the family of α-mollified values on $\mathcal{G}^N_{nz}$.

4.5 Consistency

It is shown that the affine combinations of the ED and ESD values can be characterized using projection consistency by van den Brink et al. (2016). Zou et al. (2021) use projection consistency to characterize the PD value (also see Chapter 2). Several values satisfy projection consistency, see, e.g., Otten (1993), Funaki and Yamato (2001), Calleja and Llerena (2017), and Calleja and Llerena (2019). However, we show that the only α-mollified values that satisfy projection consistency are the PD value and affine combinations of the ED and ESD values.

We refer the definitions of the projection reduced game and projection consistency to Definition 2.2 and Definition 2.3.

Consistency is usually applied together with some properties to characterize a unique value on a class of games. Instead, the following theorem uses the consistency principle to select specific values from the family of α-mollified values (and thus implicitly assumes properties that characterize this family in Theorem 4.2).

Theorem 4.3. *Let ψ^{α} on $\mathcal{G}_{nz}$ be an α-mollified value. Then ψ^{α} satisfies projection consistency if and only if $\psi^{\alpha} = PD$ or $\psi^{\alpha} = \beta ESD + (1-\beta)ED$, where $\beta \in \mathbb{R}$.*

Although we can give a direct proof of this theorem, it follows as a corollary from the following theorem using a class of values containing the α-mollified values. The difference is that in defining the linear functions $\alpha(N,v)$, we allow that coalitions of the same size are assigned a different weight by the function $\alpha(N,v)$.

Definition 4.2. Let $\alpha : \mathcal{C} \to \mathbb{R}$ be a linear function on $\mathcal{C} \subseteq \mathcal{G}_{nz}$. That is, for any $(N,v) \in \mathcal{C}$, $\alpha(N,v) = \sum\limits_{S \subseteq N} \alpha_S^N v(S)$, where every parameter $\alpha_S^N \in \mathbb{R}$. The *$\alpha$-mollified generalized value* on $\mathcal{C}$ is defined for all $(N,v) \in \mathcal{C}$ and $i \in N$, by

$$\psi_i^{\alpha}(N,v) = \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} \alpha(N,v) + \frac{1}{n}\big(v(N) - \alpha(N,v)\big). \tag{4.6}$$

Notice that Expression (4.6) is the same as Expression (4.2), but we apply a more general function $\alpha(N,v)$. It turns out that Theorem 4.3 holds on the larger class of α-mollified generalized values.

Theorem 4.4. *Let ψ^{α} on $\mathcal{G}_{nz}$ be an α-mollified generalized value. Then ψ^{α} satisfies projection consistency if and only if $\psi^{\alpha} = PD$ or $\psi^{\alpha} = \beta ESD + (1-\beta)ED$, where $\beta \in \mathbb{R}$.*

As mentioned before, the family of α-mollified values is a subfamily of α-mollified generalized values. Since the resulting values in Theorem 4.4 are also members in the family of α-mollified values, we obtain Theorem 4.3 as a corollary of Theorem 4.4.

Remark 4.4. The Core, one of the most significant set-valued solutions, has been axiomatised in terms of consistency in, e.g., Funaki and Yamato (2001) and Abe (2018). The Core is a convex set of payoff vectors for every game (if core is nonempty), whereas for a game $(N,v) \in \mathcal{G}_{nz}$, the resulting subfamily in Theorem 4.4 in general is not a convex set. As an implication of Theorem 4.4, the combinations of the PD value and any value belonging to $\{\gamma ESD + (1-\gamma)ED \mid \gamma \in \mathbb{R}\}$, such as the value given by (4.1) with $\beta \in (0,1)$, cannot be characterized using projection consistency.

Combining Theorem 4.2 and Theorem 4.3, we can obtain a result that a value ψ on $\mathcal{G}_{nz+}^N$ satisfies efficiency, the balanced individual excess ratio property, continuity, weak additivity, anonymity, no advantageous reallocation across individuals, and projection consistency if and only if ψ is the PD value or given by $\beta ESD + (1-\beta)ED$, where $\beta \in \mathbb{R}$. It is worth to mention that, with the class of α-mollified (generalized) values, these two very different types of values are obtained by projection consistency.

Moulin gives an interesting result (Theorem 2, Moulin (1987)) which characterizes the equal sharing and proportional sharing rules by separability, no advantageous reallocation, path independence and additivity in surplus sharing problems. Though our TU-game setting is different from surplus sharing problem, the projection consistency seems to have an important role to pick up the values similar to the two types of sharing rules in Moulin (1987). In this sense, our theorem can be considered as a counter part of Moulin's result.

4.6 Procedural implementation

As mentioned in Section 4.4, a dynamic process of allocating the attainable worth is another approach to justify values for TU-games; we refer to Ju et al. (2007b), Hwang et al. (2005), Malawski (2013), Wang et al. (2019), and Hou et al. (2019). Under the assumption that all players form the grand coalition, as is usual in the theory of TU-games, the players then totally allocate the worth of the grand coalition among themselves. Given a formation order, a player claims her share in the worth of the grand coalition when she joins the game, and what's left is allocated among the players who have arrived before him. Motivated by this procedure, we give a procedural implementation of the α-mollified values based on a one-by-one formation of the grand coalition.

Formally, a unique payoff vector is determined by the following steps:

Step 1: Choose any game $(N, v) \in \mathcal{G}^N_{nz}$ and any permutation $\pi \in \Pi(N)$ to gradually form the grand coalition N.

Step 2: Each entering player $i \in N$ such that $\pi(i) = 1$ receives his individual worth $v(\{i\})$.

Step 3: Each entering player $i \in N$ such that $\pi(i) \neq 1$ obtains the fraction $\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}$ from the preservated worth $\alpha(N, v)$.

Step 4: The residual $v(S^i_\pi) - v(S^i_\pi \backslash \{i\}) - \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\alpha(N, v)$ after player i joining the nonempty coalition $S^i_\pi \setminus \{i\}$, $S^i_\pi = \{j \in N \mid \pi(j) \leq \pi(i)\}$, is allocated equally among the members of the coalition that was present before i entered.

Steps 1–4 determine a payoff vector $(\eta^{\pi,\alpha})_{i\in N} \in \mathbb{R}^N$ defined as:

$$\eta_i^{\pi,\alpha} = \begin{cases} v(\{i\}) + \sum\limits_{\substack{j\in N:\\ \pi(j)>\pi(i)}} \frac{v(S^j_\pi) - v(S^j_\pi \backslash \{j\}) - \frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{\pi(j)-1}, & \text{if } \pi(i) = 1, \\ \frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})} + \sum\limits_{\substack{j\in N:\\ \pi(j)>\pi(i)}} \frac{v(S^j_\pi) - v(S^j_\pi \backslash \{j\}) - \frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{\pi(j)-1}, & \text{if } \pi(i) \neq 1, n, \\ \frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}, & \text{if } \pi(i) = n. \end{cases} \tag{4.7}$$

The next theorem shows that averaging the outcome of this procedure over all permutations yields the corresponding α-mollified value. In fact, by the same proof it can be shown that this holds for any α-mollified generalized value.

Theorem 4.5. *For any $(N,v) \in \mathcal{G}^N_{nz}$ and any linear function $\alpha(N,v) = \sum_{S \subseteq N} \alpha^{|N|}_{|S|} v(S)$,*

$$\psi^{\alpha}_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \eta^{\pi,\alpha}_i \quad \textit{for all } i \in N,$$

where $\eta^{\pi,\alpha}_i$ is given by (4.7).

Again, the proof can be found in Section 4.8.

Remark 4.5. Theorem 4.5 is still valid for games in which the sum of all stand-alone worths is nonzero. For the procedural implementation, the number $\frac{v(\{i\})}{\sum_{k \in N} v(\{k\})}$ for each $i \in N$ can be considered as an endogenous weight of player i. If this endogenous weight vector is replaced by an exogenous weight vector $(w_i)_{i \in N}$, i.e. $w_i \geq 0$ for all $i \in N$ and $\sum_{k \in N} w_k = 1$, then the expected payoff is given by

$$\psi^{w}_i(N,v) = w_i \alpha(N,v) + \frac{1}{n}\big(v(N) - \alpha(N,v)\big) \quad \text{for all } i \in N.$$

4.7 Independence of axioms

Logical independence of the axioms in Theorem 4.2 can be shown by the following alternative values on $\mathcal{G}^N_{nz+}$ (or $\mathcal{G}^N_{nz-}$):

(i) The value defined by $\psi_i(N,v) = 0$ for all $i \in N$ satisfies all axioms except efficiency.

(ii) The value defined by

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{S \subseteq N : i \in S, |S| \geq 2} v(S) - \frac{1}{n} \sum_{S \subseteq N : |S| \geq 2} |S| v(S) \quad \text{for all } i \in N,$$

satisfies all axioms except the balanced individual excess ratio property.

(iii) Let $\widehat{f} : \mathbb{R}^{2^n - n - 1} \to \mathbb{R}$ be a discontinuous additive function[3] with respect to all $v(S)$, $S \subseteq N$, $|S| \geq 2$. The value defined by

$$\psi_i(N,v) = \frac{v(N)}{n} + \widehat{f}(N,v)\left(\frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} - \frac{1}{n}\right) \quad \text{for all } i \in N,$$

satisfies all axioms except continuity.

(iv) The value defined by

$$\psi_i(N,v) = \frac{v(N)}{n} + \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} - \frac{1}{n} \quad \text{for all } i \in N,$$

[3]The existence of discontinuous additive functions was an open problem for many years. Mathematicians could neither prove that every additive function is continuous nor exhibit an example of a discontinuous additive function. It was Hamel (1905) who first succeeded in proving that there exist discontinuous additive functions. No concrete example is known, but we only know that it exists; we refer to (pp.129-130, Kuczma (2009)) and (pp.9-13, Sahoo and Kannappan (2011)).

satisfies all axioms except weak additivity.

(v) Let $\{\alpha_S^N \mid S \subseteq N, |S| \geq 2\}$ be a collection of real numbers such that $\alpha_T^N \neq \alpha_K^N$ for some $T, K \subseteq N$ with $|T| = |K|$. The value defined by

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{S \subseteq N : |S| \geq 2} \alpha_S^N v(S) \left(\frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} - \frac{1}{n} \right) \quad \text{for all } i \in N,$$

satisfies all axioms except anonymity.

(vi) The value defined by

$$\psi_i(N,v) = \frac{v(N)}{n} + \frac{\sum_{k \in N} (v(\{k\}))^2}{\sum_{k \in N} v(\{k\})} \left(\frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} - \frac{1}{n} \right) \quad \text{for all } i \in N,$$

satisfies all axioms except no advantageous reallocation across individuals.

4.8 Proofs

Let us denote $K(v) = \sum_{i \in N} v(\{i\})$ for any game (N,v). If no ambiguity is possible, we use K instead of $K(v)$.

Proof of Proposition 4.1. It is clear that the 'if' part is satisfied. To show the 'only if' part, suppose that ψ is a value satisfying efficiency, the balanced individual excess ratio property, and continuity. We distinguish the games of $\mathcal{G}_{nz}^N$ into two cases.

Case (i): $(N,v) \in \mathcal{G}_D^N$. Notice that there being some pair of players $i, j \in N$ with $v(\{i\}) \neq v(\{j\})$, implies that for any $i \in N$, there must be some player $j \in N$ such that $v(\{i\}) \neq v(\{j\})$. The balanced individual excess ratio property implies that for all $k \in N \backslash \{i\}$,

$$(\psi_i(N,v) - \psi_j(N,v))(v(\{i\}) - v(\{k\})) = (\psi_i(N,v) - \psi_k(N,v))(v(\{i\}) - v(\{j\})).$$

Denoting $g_i(N,v) = \frac{\psi_i(N,v) - \psi_j(N,v)}{v(\{i\}) - v(\{j\})}$, the above equation can be written as

$$\psi_i(N,v) - \psi_k(N,v) = g_i(N,v)[v(\{i\}) - v(\{k\})].$$

Summing this equality over all $k \in N \backslash \{i\}$, yields

$$(n-1)\psi_i(N,v) - \sum_{k \in N \backslash \{i\}} \psi_k(N,v) = g_i(N,v)[(n-1)v(\{i\}) - \sum_{k \in N \backslash \{i\}} v(\{k\})],$$

which can be rewritten as

$$n\psi_i(N,v) - \sum_{k \in N} \psi_k(N,v) = g_i(N,v)[nv(\{i\}) - \sum_{k \in N} v(\{k\})].$$

With efficiency, $\sum_{k\in N}\psi_k(N,v)=v(N)$, and thus this implies

$$\psi_i(N,v)=\frac{v(N)}{n}+g_i(N,v)[v(\{i\})-\frac{1}{n}\sum_{k\in N}v(\{k\})]. \tag{4.8}$$

Notice that, for every player i in a game (N,v), the number $g_i(N,v)$ is determined for a specific j, but this j can be different for different games (as long as it is a player with a different stand-alone worth as i.) Next, we show that $g_i(N,v)=g_h(N,v)$ for all $i,h\in N$. Let $i,h\in N$ be two players such that $v(\{i\})\neq v(\{h\})$. Clearly, $g_i(N,v)=g_h(N,v)=\frac{\psi_i(N,v)-\psi_h(N,v)}{v(\{i\})-v(\{h\})}$. For any $k\in N\backslash\{i,h\}$, it must be that $v(\{k\})\neq v(\{i\})$ or $v(\{k\})\neq v(\{h\})$ (or both). Without loss of generality, we assume that $v(\{k\})\neq v(\{i\})$. By the balanced individual excess ratio property applied to $i,h,k\in N$, we have $\frac{\psi_i(N,v)-\psi_h(N,v)}{v(\{i\})-v(\{h\})}=\frac{\psi_i(N,v)-\psi_k(N,v)}{v(\{i\})-v(\{k\})}$, implying $g_i(N,v)=g_k(N,v)$.

Setting $g(N,v)=g_i(N,v)$ and then substituting it into (4.8), we obtain (4.4). Moreover, continuity implies that $g(N,v)$ is continuous.

Case (ii): $(N,v)\in\mathcal{G}^N_{nz}\setminus\mathcal{G}^N_D$. In this case, $v(\{i\})=v(\{j\})$ for all $i,j\in N$. Let $\{(N,w_m)\}$ be a sequence of games from $\mathcal{G}^N_D$ such that $\lim_{m\to\infty}(N,w_m)=(N,v)$. By continuity and Case (i),

$$\begin{aligned}\psi_i(N,v)&=\lim_{m\to\infty}\psi_i(N,w_m)\\&=\lim_{m\to\infty}[\frac{v(N)}{n}+g(N,w_m)[w_m(\{i\})-\frac{1}{n}\sum_{k\in N}w_m(\{k\})]]\\&=\frac{v(N)}{n}+\lim_{m\to\infty}g(N,w_m)[w_m(\{i\})-\frac{1}{n}\sum_{k\in N}w_m(\{k\})]\\&=\frac{v(N)}{n},\end{aligned}$$

where the last equality follows from the fact that $g(N,w_m)$ is a continuous function and, by $\lim_{m\to\infty}(N,w_m)=(N,v)$ with $(N,v)\notin\mathcal{G}^N_D$, $\lim_{m\to\infty}[w_m(\{i\})-\frac{1}{n}\sum_{k\in N}w_m(\{k\})]=0$. Clearly, this coincides with (4.4), for any function $g(N,v)$. Taking $g(N,v)=\lim_{m\to\infty}g(N,w_m)$ yields the desired assertion.

□

Proof of Theorem 4.1. It is clear that any value of the form given in (4.5) satisfies the three axioms. To prove the 'only if' part, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying the three axioms. The proof is divided into three steps. Step 1 and Step 2 together show that (4.5) holds on the class $\mathcal{G}^N_D$, and Step 3 shows that (4.5) also holds on the class $\mathcal{G}^N_{nz}\backslash\mathcal{G}^N_D$.

Step 1. For any $(N,v)\in\mathcal{G}^N_D$, there exists two games $(N,v_1),(N,v_2)\in\mathcal{G}^N_D$ such that $v=v_1+v_2$. We show a relationship among the payoffs of three games (N,v), (N,v_1) and (N,v_2). From case (i) of the proof of Proposition 4.1, efficiency and the balanced individual excess ratio property imply that ψ on $\mathcal{G}_D^N$ has the form given in (4.4), but g is not guaranteed to be continuous. By linearity, we have $\psi_h(N,v_1) + \psi_h(N,v_2) = \psi_h(N,v)$ for all $h \in N$. This yields

$$
\begin{aligned}
&g(N,v_1)[v_1(\{h\}) - \frac{1}{n}K(v_1)] + g(N,v_2)[v_2(\{h\}) - \frac{1}{n}K(v_2)] \\
=&g(N,v)[v(\{h\}) - \frac{1}{n}K(v)] \\
=&g(N,v)[v_1(\{h\}) + v_2(\{h\}) - \frac{1}{n}K(v_1) - \frac{1}{n}K(v_2)].
\end{aligned}
$$

Denote $x = g(N,v) - g(N,v_1)$ and $y = g(N,v) - g(N,v_2)$. The above equation can then be rewritten as

$$x[v_1(\{h\}) - \frac{1}{n}K(v_1)] + y[v_2(\{h\}) - \frac{1}{n}K(v_2)] = 0.$$

Subtracting these equations for any two distinct players $h, l \in N$ yields

$$x[v_1(\{h\}) - v_1(\{l\})] + y[v_2(\{h\}) - v_2(\{l\})] = 0. \tag{4.9}$$

Step 2. Next, using (4.9) we show that g is a constant function on $\mathcal{G}_D^N$. We remark that we only have to consider the individually positive games, since $(N,v) \in \mathcal{G}_D^N$ implies $(N,-v) \in \mathcal{G}_D^N$. By (4.4) applied to $(N,-v)$, we have

$$
\begin{aligned}
\psi_i(N,-v) &= \frac{-v(N)}{n} + g(N,-v)[-v(\{i\}) + \frac{1}{n}\sum_{k\in N} v(\{k\})] \\
&= -[\frac{v(N)}{n} + g(N,-v)[v(\{i\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})]].
\end{aligned}
$$

By linearity, $\psi_i(N,-v) = -\psi_i(N,v)$. Taking into account the above equation and (4.4), we obtain $g(N,v) = g(N,-v)$.

Let $(N,v),(N,w) \in \mathcal{G}_D^N \cap \mathcal{G}_{nz+}^N$ and $i,j \in N$ be such that $v(\{i\}) \neq v(\{j\})$. Clearly, there must be a player $k \in N$ such that $w(\{k\}) \neq w(\{i\})$ or $w(\{k\}) \neq w(\{j\})$. Without loss of generality, we assume that $w(\{i\}) \neq w(\{k\})$ for a given $k \in N \backslash \{i,j\}$. To show that g is a constant, we consider two cases:

Case (i): Suppose that $v(\{i\}) - v(\{j\}) \neq w(\{i\}) - w(\{j\})$ and $v(\{i\}) - v(\{k\}) \neq w(\{i\}) - w(\{k\})$. Denote $\varepsilon = \sum_{h\in\{i,j,k\}}[v(\{h\}) + w(\{h\})]$. We define the following three games.

$$
u(S) = \begin{cases}
\varepsilon, & \text{if } S = \{i\}, \\
v(\{j\}) - v(\{i\}) + \varepsilon, & \text{if } S = \{j\}, \\
w(\{k\}) - w(\{i\}) + \varepsilon, & \text{if } S = \{k\}, \\
v(\{h\}) + w(\{h\}), & \text{if } S = \{h\}, h \in N\backslash\{i,j,k\}. \\
0, & \text{if } S \subseteq N \text{ with } |S| \geq 2.
\end{cases}
$$

$$v_0(S) = \begin{cases} v(\{i\}) - \varepsilon, & \text{if } S = \{h\}, h \in \{i,j\}, \\ v(\{k\}) - w(\{k\}) + w(\{i\}) - \varepsilon, & \text{if } S = \{k\}, \\ -w(\{h\}), & \text{if } S = \{h\}, h \in N \backslash \{i,j,k\}. \\ v(S), & \text{if } S \subseteq N \text{ with } |S| \geq 2. \end{cases}$$

$$w_0(S) = \begin{cases} w(\{i\}) - \varepsilon, & \text{if } S = \{h\}, h \in \{i,k\}, \\ w(\{j\}) - v(\{j\}) + v(\{i\}) - \varepsilon, & \text{if } S = \{j\}, \\ -v(\{h\}), & \text{if } S = \{h\}, h \in N \backslash \{i,j,k\}. \\ w(S), & \text{if } S \subseteq N \text{ with } |S| \geq 2. \end{cases}$$

Clearly, (i) (N,u) is individually positive with $u(\{i\}) \neq u(\{j\})$ and $u(\{i\}) \neq u(\{k\})$; (ii) (N,v_0) is individually negative with $v_0(\{i\}) \neq v_0(\{k\})$; (iii) (N,w_0) is individually negative with $w_0(\{i\}) \neq w_0(\{j\})$. Moreover, $(N,u) + (N,v_0) = (N,v)$ and $(N,u) + (N,w_0) = (N,w)$.

By (4.9) applied to (N,u), (N,v_0), (N,v), and players i,j, we have

$$x[u(\{i\}) - u(\{j\})] + y[v_0(\{i\}) - v_0(\{j\})] = 0.$$

Since $u(\{i\}) - u(\{j\}) \neq 0$ and $v_0(\{i\}) - v_0(\{j\}) = 0$, then

$$x = g(N,v) - g(N,u) = 0. \tag{4.10}$$

Similarly, by (4.9) applied to (N,u), (N,w_0), (N,w), and players i,k, we have

$$g(N,w) - g(N,u) = 0. \tag{4.11}$$

Together, (4.10) and (4.11) imply

$$g(N,v) = g(N,w). \tag{4.12}$$

Case (ii): Suppose that $v(\{i\}) - v(\{j\}) = w(\{i\}) - w(\{j\})$ or/and $v(\{i\}) - v(\{k\}) = w(\{i\}) - w(\{k\})$. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}_+$ be such that $\varepsilon_3 < \min\{v(\{i\}), v(\{j\})\} + \varepsilon_1$. Consider the following games.

$$v_3(S) = \begin{cases} v(S) + \varepsilon_1, & \text{if } S = \{h\}, h \in \{i,j\}, \\ v(S) + \varepsilon_2, & \text{if } S = \{h\}, h \in N \backslash \{i,j\}. \\ v(S), & \text{if } S \subseteq N \text{ with } |S| \geq 2. \end{cases}$$

$$v_4(S) = \begin{cases} -\varepsilon_1, & \text{if } S = \{h\}, h \in \{i,j\}, \\ -\varepsilon_2, & \text{if } S = \{h\}, h \in N \backslash \{i,j\}. \\ 0, & \text{if } S \subseteq N \text{ with } |S| \geq 2. \end{cases}$$

$$v_5(S) = \begin{cases} v(S) + \varepsilon_1 - \varepsilon_3, & \text{if } S = \{h\}, h \in \{i,j\}, \\ v(S), & \text{otherwise.} \end{cases}$$

$$v_6(S) = \begin{cases} \varepsilon_3, & \text{if } S = \{h\}, h \in \{i,j\}, \\ \varepsilon_2, & \text{if } S = \{h\}, h \in N\backslash\{i,j\}. \\ 0, & \text{if } S \subseteq N \text{ with } |S| \geq 2. \end{cases}$$

Notice that (N, v_3), (N, v_5) and (N, v_6) are individually positive games, and (N, v_4) is an individually negative game. Clearly, $(N, v) = (N, v_3) + (N, v_4)$ and $(N, v_3) = (N, v_5) + (N, v_6)$. By (4.9), we have $x(v_3(\{i\}) - v_3(\{j\})) + y(v_4(\{i\}) - v_4(\{j\})) = 0$, and since $v_3(\{i\}) \neq v_3(\{j\})$ and $v_4(\{i\}) = v_4(\{j\})$, we have $x = g(N, v) - g(N, v_3) = 0$. Similar, $x(v_5(\{i\}) - v_5(\{j\})) + y(v_6(\{i\}) - v_6(\{j\})) = 0$, and since $v_5(\{i\}) \neq v_5(\{j\})$ and $v_6(\{i\}) = v_6(\{j\})$, we have $x = g(N, v_3) - g(N, v_5) = 0$. Thus,

$$g(N, v) = g(N, v_5), \tag{4.13}$$

showing that g remains unchanged if two different stand-alone worths change by the same amount, and the new game still is a member of $\mathcal{G}^N_{nz}$.

With (4.13), we can construct a game $(N, v') \in \mathcal{G}^N_D$ such that $g(N, v') = g(N, v)$, $v'(\{i\}) \neq v'(\{j\})$, $v'(\{i\}) - v'(\{j\}) \neq w(\{i\}) - w(\{j\})$ and $v'(\{i\}) - v'(\{k\}) \neq w(\{i\}) - w(\{k\})$.[4]

Since (N, v') is as in Case (i), from Case (i) we have $g(N, v') = g(N, w)$. Thus, $g(N, v) = g(N, w)$.

Cases (i) and (ii) show that $g(N, v) = g(N, w)$ for all $(N, v), (N, w) \in \mathcal{G}^N_D$, and thus g is a constant function on $\mathcal{G}^N_D$. Setting $\beta = g(N, v)$, we obtain (4.5) on the class $\mathcal{G}^N_D$.

Step 3. Finally, we show that g is a constant function on $\mathcal{G}^N_{nz}$, and thus (4.5) holds on $\mathcal{G}^N_{nz}$. From Step 2, we obtain (4.5) only on the class $\mathcal{G}^N_D$. On the other hand, consider any game $(N, v) \in \mathcal{G}^N_{nz} \backslash \mathcal{G}^N_D$ such that $v(\{i\}) = v(\{j\})$ for all $i, j \in N$. Obviously, there exists two games $(N, v_1), (N, v_2) \in \mathcal{G}^N_D$ such that $(N, v) = (N, v_1) + (N, v_2)$. We obtain from Step 2 that $g(N, v_1) = g(N, v_2) = \beta$, and (4.5) holds for (N, v_1) and (N, v_2). By linearity and (4.5) applied to (N, v_1) and (N, v_2), we have

$$\begin{aligned} \psi_i(N, v) &= \psi_i(N, v_1) + \psi_i(N, v_2) \\ &= \frac{v_1(N)}{n} + \beta[v_1(\{i\}) - \frac{1}{n} \sum_{k \in N} v_1(\{k\})] \end{aligned}$$

[4] To illustrate this for the case that $v(\{i\}) - v(\{j\}) = w(\{i\}) - w(\{j\})$ and $v(\{i\}) - v(\{k\}) = w(\{i\}) - w(\{k\})$ (the case that only one of these equalities is satisfied goes similar), we can construct the game (N, v') given by $v'(\{i\}) = v(\{i\}) + 2v(\{k\})$, $v'(\{j\}) = v(\{j\}) + 2v(\{k\}) + v(\{i\})$, $v'(\{k\}) = v(\{k\}) + v(\{i\})$ and $v'(S) = v(S)$ otherwise. Obviously, $v'(\{i\}) \neq v'(\{j\})$, $v'(\{i\}) - v'(\{j\}) = -v(\{j\}) \neq w(\{i\}) - w(\{j\})$ and $v'(\{i\}) - v'(\{k\}) = v(\{k\}) \neq w(\{i\}) - w(\{k\})$. Going from (N, v) to (N, v') in two steps ($[v(\{i\}), v(\{j\}), v(\{k\})] \rightarrow [v(\{i\}) + 2v(\{k\}), v(\{j\}) + 2v(\{k\}), v(\{k\})] \rightarrow [v(\{i\}) + 2v(\{k\}), v(\{j\}) + 2v(\{k\}) + v(\{i\}), v(\{k\}) + v(\{i\})]$, we can twice apply that the value of function g does not change, and thus $g(N, v) = g(N, v')$.

$$
\begin{aligned}
&+\frac{v_2(N)}{n}+\beta[v_2(\{i\})-\frac{1}{n}\sum_{k\in N}v_2(\{k\})]\\
&=\frac{v_1(N)+v_2(N)}{n}+\beta[v_1(\{i\})+v_2(\{i\})-\frac{1}{n}\sum_{k\in N}(v_1(\{k\})+v_2(\{k\}))]\\
&=\frac{v(N)}{n}+\beta[v(\{i\})-\frac{1}{n}\sum_{k\in N}v(\{k\})]\\
&=\frac{v(N)}{n},
\end{aligned}
$$

where the last equality holds since $v(\{i\}) = v(\{j\})$ for all $i, j \in N$. This is the ED value for any $\beta \in \mathbb{R}$, and coincides with (4.1) if $v(\{i\}) = v(\{j\})$ for all $i, j \in N$. □

Proof of Theorem 4.2

Before presenting the lengthy proof of Theorem 4.2, we introduce two theorems on Cauchy functional equations.

Theorem 4.6. *(see, Theorem 5.5.2, p.139, Kuczma, 2009) If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous additive function, then there exists $d \in \mathbb{R}^n$ such that $f(x) = \sum_{i\in N} d_i x_i$.*

For $x, y \in \mathbb{R}^n$, let $x - y = (x_1 - y_2, x_2 - y_2, \ldots, x_n - y_n)$.

Theorem 4.7. *Let $f : \mathbb{R}_+ \times \mathbb{R}^{n-1} \to \mathbb{R}$ be an additive function. Then $F : \mathbb{R}^n \to \mathbb{R}$ given by*

$$F(x-y) = f(x) - f(y) \quad \text{for all } x, y \in \mathbb{R}_+ \times \mathbb{R}^{n-1}, \tag{4.14}$$

is an additive function such that $F(x) = f(x)$ for all $x \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$.

Proof. First, we show that F is well-defined (that is, the function given in (4.14) is a valid definition of a function). Let $x, y, h, l \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ be such that $x - h = y - l$. Hence, $x + l = y + h \Rightarrow f(x + l) = f(y + h) \Rightarrow f(x) + f(l) = f(y) + f(h) \Rightarrow f(x) - f(h) = f(y) - f(l) \Rightarrow F(x - h) = F(y - l)$, where the second implication follows from additivity of f.

Next, we show that F is an extension of f. For any $t \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, there exist $x, y \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ such that $t = x - y$. Hence, $F(t) = F(x - y) = f(x) - f(y) = f(y + t) - f(y) = f(y) + f(t) - f(y) = f(t)$, as asserted.

Finally, we show that F is additive on $\mathbb{R}^n$. For any $s, t \in \mathbb{R}^n$, there exist $x, y, h, l \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ such that $s = x - h$ and $t = y - l$. Note that $x + y$ and $t + l$ are in $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. Also, $s + t = (x + y) - (h + l)$. Then, by (4.14), we have $F(s + t) = F((x + y) - (h + l)) = f(x + y) - f(h + l) = f(x) + f(y) - f(h) - f(l) = F(x - h) + F(y - l) = F(s) + F(t)$, as asserted, where the third equality follows from additivity of f. □

Theorem 4.7 is a modification of Theorem 2 in Aczél and Erdős (1965) or Lemma 6.2 in Reem (2017) that provides an extensive principle on conditional Cauchy equation[5]. A similar theorem (with a similar proof) holds if $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ is replaced by $\mathbb{R}_- \times \mathbb{R}^{n-1}$.

Next, we give additional notation and a remark. For any $c = (c_i)_{i\in N} \in \mathbb{R}^N_+$, the classes $\mathcal{G}^N_{c+}$ and $\mathcal{G}^N_{c-}$ are denoted as follows.

$$\mathcal{G}^N_{c+} = \{(N,v) \in \mathcal{G}^N_{nz} \mid \exists a \in \mathbb{R}_+ : \forall\, i \in N, v(\{i\}) = ac_i\}.$$
$$\mathcal{G}^N_{c-} = \{(N,v) \in \mathcal{G}^N_{nz} \mid \exists a \in \mathbb{R}_- : \forall\, i \in N, v(\{i\}) = ac_i\}.$$

That is, $\mathcal{G}^N_{c+}$ (respectively $\mathcal{G}^N_{c-}$) consists of all games in $\mathcal{G}^N_{nz}$ in which the players' stand-alone worths are in the same positive (respectively negative) proportion to c.

Remark 4.6. Each game $(N,v) \in \mathcal{G}^N_{c+}$ is represented by a vector $p^v = (p^v_1, p^v_S) \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$, where the first component $p^v_1 = \frac{v(\{i\})}{c_i}$ is the ratio of $v(\{i\})$ to c_i which is equal and positive for all $i \in N$, and the remaining $2^n - n - 1$ components are the worths $v(S)$ of the $2^n - n - 1$ coalitions $S \subseteq N$, $|S| \geq 2$. Moreover, $(N,v) \in \mathcal{G}^N_{c+} \Leftrightarrow p^v \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$. Similarly, $(N,v) \in \mathcal{G}^N_{c-} \Leftrightarrow p^v \in \mathbb{R}_- \times \mathbb{R}^{2^n-n-1}$. Clearly, $(N,v) = (N,v_1) + (N,v_2) \Leftrightarrow p^v = p^{v_1} + p^{v_2}$.

Proof of Theorem 4.2. It is easily checked that any value of the form given in (4.2) satisfies the six axioms. To show the 'only if' part, let ψ be a value on $\mathcal{G}^N_{nz+}$ satisfying the six axioms. (The proof on $\mathcal{G}^N_{nz-}$ goes in a similar way.) From Remark 4.2, ψ has the form given by (4.4). Define a function $f : \mathcal{G}^N_{nz+} \to \mathbb{R}$ by $f(N,v) = K(v)g(N,v)$ for all $(N,v) \in \mathcal{G}^N_{nz+}$, where $K(v) = \sum_{i\in N} v(\{i\})$, as defined in the beginning of Section 4.8. Then (4.4) can be rewritten as

$$\psi_i(N,v) = \frac{v(N)}{n} + f(N,v)\left(\frac{v(\{i\})}{K(v)} - \frac{1}{n}\right). \tag{4.15}$$

Clearly, $f(N,v)$ is continuous since $K(v)$ and $g(N,v)$ are continuous.

We will consider five steps to show the rest of the 'only if' part. Step 1 formulates the value on $\mathcal{G}^N_{c+}$ that satisfies efficiency, the balanced individual excess ratio property, continuity, and weak additivity. Using no advantageous reallocation across individuals, Step 2 and Step 3 derive the coefficients of the formula obtained in Step 1 if $c \in \mathbb{R}^N_+$ is such that $c_i \neq c_j$ for some $i,j \in N$. Step 4 considers the case that $c \in \mathbb{R}^N_+$ with $c_i = c_j$ for all $i,j \in N$. Step 5 gives the desired formula by anonymity.

Step 1. Pick any $c \in \mathbb{R}^N_+$. We derive the formula of ψ on $\mathcal{G}^N_{c+}$ satisfying efficiency, the balanced individual excess ratio property, continuity, and weak additivity. Consider $(N,v),(N,w) \in \mathcal{G}^N_{c+}$. Since $(N,v+w) \in \mathcal{G}^N_{c+}$, weak additivity implies that

[5] The Cauchy functional equation is a well-known and fundamental equation in the theory of functional equations. It is given by $f(x+y) = f(x) + f(y)$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. A conditional Cauchy equation is a variation of this equation by changing the domain of validity of the equation.

$\psi_i(N, v+w) = \psi_i(N,v) + \psi_i(N,w)$ for all $i \in N$. Taking (4.15) into account, we obtain

$$f(N, v+w) = f(N,v) + f(N,w). \tag{4.16}$$

Let $h : \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1} \to \mathbb{R}$ be defined by $h(p^v) = f(N,v)$ for all $p^v \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$ with $(N,v) \in \mathcal{G}^N_{c+}$ as in Remark 4.6. Then, (4.16) can be rewritten as

$$h(p^{v+w}) = h(p^v + p^w) = h(p^v) + h(p^w),$$

which shows that h is an additive function on $\mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$.

Moreover, h is continuous since f is continuous. Hence, from Theorem 4.7, there exists an additive function $H : \mathbb{R}^{2^n-n} \to \mathbb{R}$ such that $H(x) = h(x)$ for all $x \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$. Obviously, H is continuous since (i) H has the form of (4.14), i.e. $H(x-y) = h(x) - h(y)$ for all $x, y \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}$, and (ii) h is continuous. Thus, from Theorem 4.6, there exists $d \in \mathbb{R}^{2^n-n}$ such that

$$h(x) = H(x) = \sum_{i=1}^{2^n-n} d_i x_i \quad \text{for all } x \in \mathbb{R}_+ \times \mathbb{R}^{2^n-n-1}.$$

Equivalently, by definition of h,

$$f(N,v) = h(p^v) = d_1^c p_1^v + \sum_{S\subseteq N, |S|\geq 2} d_S^c v(S) \quad \text{for all } (N,v) \in \mathcal{G}^N_{c+}, \tag{4.17}$$

where d_1^c and d_S^c, $S \subseteq N$, $|S| \geq 2$, are real numbers, and $p_1^v = \frac{v(\{i\})}{c_i} \in \mathbb{R}$ for all $i \in N$ is the ratio of $v(\{i\})$ to c_i. Clearly, these coefficients depend on c since $(N,v) \in \mathcal{G}^N_{c+}$.

Taking into account that $d_1^c = d_{\{i\}}^c \in \mathbb{R}$ and $p_1^v = \frac{v(\{i\})}{c_i}$ for each $i \in N$, from (4.17), we obtain a system of n linearly independent equations. Summing these equations over all players, we obtain

$$nf(N,v) = \sum_{i\in N} \frac{d_{\{i\}}^c}{c_i} v(\{i\}) + n \sum_{S\subseteq N, |S|\geq 2} d_S^c v(S). \tag{4.18}$$

Since $\mathcal{G}^N_{c+} = \mathcal{G}^N_{\beta c+}$ for all $\beta \in \mathbb{R}_+$, then $(N,v) \in \mathcal{G}^N_{c+}$ implies $(N,v) \in \mathcal{G}^N_{\beta c+}$. Thus, similar to (4.18), we have

$$nf(N,v) = \sum_{i\in N} \frac{d_{\{i\}}^{\beta c}}{\beta c_i} v(\{i\}) + n \sum_{S\subseteq N, |S|\geq 2} d_S^{\beta c} v(S). \tag{4.19}$$

Notice that the left-hand side of (4.18) is identical to that of (4.19), which yields

$$\frac{d_{\{i\}}^c}{c_i} = \frac{d_{\{i\}}^{\beta c}}{\beta c_i}, \quad \text{and } d_S^c = d_S^{\beta c} \text{ for each } S \subseteq N \text{ with } |S| \geq 2.$$

Therefore, setting $\alpha^c_{\{i\}} := \frac{d^c_{\{i\}}}{nc_i}$ and $\alpha^c_S := d^c_S$, (4.18) can be rewritten as

$$f(N,v) = \sum_{S\subseteq N} \alpha^c_S v(S). \tag{4.20}$$

Notice that α^c_S is a real number depending on S and c, and, moreover, for each $S \subseteq N$ with $|S| \geq 2$ and all $\beta \in \mathbb{R}_+$,

$$\alpha^c_S = \alpha^{\beta c}_S. \tag{4.21}$$

Substituting (4.20) into (4.15), we obtain

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{S\subseteq N} \alpha^c_S v(S)\left(\frac{v(\{i\})}{K(v)} - \frac{1}{n}\right). \tag{4.22}$$

Step 2. We show that in (4.22), $\alpha^c_S = \alpha^{c'}_S$ for each $S \subseteq N$ with $|S| \geq 2$, and thus α^c_S, $S \subseteq N$, $|S| \geq 2$, does not depend on $c \in \mathbb{R}^N_+$, if $c_i \neq c_j$ for some $i, j \in N$.

With (4.21), we only need to consider any $c, c' \in \mathbb{R}^N_+$ with $\sum_{k\in N} c_k = \sum_{k\in N} c'_k$. Let $i \in N$ be a player such that $c_i = \min\{c_k \mid k \in N\} < \frac{\sum_{k\in N} c_k}{n}$. Suppose that there exists $j \in N$ such that $c_i \neq c_j$. Without loss of generality, we assume that $c'_j \leq \frac{\sum_{k\in N} c'_k}{2}$. (If $c'_j > \frac{\sum_{k\in N} c'_k}{2}$, we pick $h \in N\backslash\{i,j\}$ and define $c^* \in \mathbb{R}^N_+$ such that $c^*_i = c_i$, $c^*_h = c'_h$, and $\sum_{k\in N} c^*_k = \sum_{k\in N} c_k$. The proof then goes in a similar way.) Then there exists a $c^* \in \mathbb{R}^N_+$ such that $c^*_i = c_i$, $c^*_j = c'_j$, and $\sum_{k\in N} c^*_k = \sum_{k\in N} c_k$. Let $(N,v),(N,w) \in \mathcal{G}^N_{nz+}$ be two games such that $v(\{k\}) = c_k$, $w(\{k\}) = c^*_k$ for all $k \in N$, and $v(S) = w(S)$ for all $S \subseteq N$ with $|S| \geq 2$. Since $v(\{i\}) = w(\{i\})$ and $\sum_{k\in N\backslash\{i\}} v(\{k\}) = \sum_{k\in N\backslash\{i\}} w(\{k\})$, no advantageous reallocation across individuals implies

$$\sum_{k\in N\backslash\{i\}} \psi_k(N,v) = \sum_{k\in N\backslash\{i\}} \psi_k(N,w).$$

With efficiency, this implies

$$\psi_i(N,v) = \psi_i(N,w).$$

Taking (4.22) into account, we obtain with $v(\{i\}) = w(\{i\})$ and $K(v) = K(w)$ that

$$\begin{aligned}
&\sum_{S\subseteq N} \alpha^c_S v(S) = \sum_{S\subseteq N} \alpha^{c^*}_S w(S)\\
\Leftrightarrow &\sum_{k\in N} \alpha^c_{\{k\}} v(\{k\}) + \sum_{S\subseteq N:|S|\geq 2} \alpha^c_S v(S) = \sum_{k\in N} \alpha^{c^*}_{\{k\}} w(\{k\}) + \sum_{S\subseteq N:|S|\geq 2} \alpha^{c^*}_S w(S)\\
\Leftrightarrow &\sum_{k\in N} \alpha^c_{\{k\}} v(\{k\}) + \sum_{S\subseteq N:|S|\geq 2} \alpha^c_S v(S) = \sum_{k\in N} \alpha^{c^*}_{\{k\}} w(\{k\}) + \sum_{S\subseteq N:|S|\geq 2} \alpha^{c^*}_S v(S).
\end{aligned} \tag{4.23}$$

Pick any $T \subseteq N$ with $|T| \geq 2$. Let $(N,v_1),(N,w_1) \in \mathcal{G}^N_{nz+}$ be such that $v_1(T) = w_1(T) \neq v(T)$, and $v_1(S) = v(S)$ and $w_1(S) = w(S)$ for all $S \subseteq N$, $S \neq T$. Similar to (4.23), we can derive an equation that the only difference with (4.23) is the terms of $v_1(T)$ and $w_1(T)$. Substituting this equation into (4.23) yields $(v(T) - v_1(T))\alpha^c_T =$

$(v(T) - w_1(T))\alpha_T^{c^*}$, which implies

$$\alpha_T^c = \alpha_T^{c^*}.$$

Similarly, consider (N, w) and (N, u), where $(N, u) \in \mathcal{G}_{nz+}^N$ is such that $u(\{k\}) = c'_k$ for all $k \in N$, and $u(S) = w(S)$ otherwise. We obtain

$$\alpha_T^{c'} = \alpha_T^{c^*},$$

where c^* is as defined before. Thus, $\alpha_T^c = \alpha_T^{c'}$ for each $T \subseteq N$ with $|T| \geq 2$.

Step 3. We show that in (4.22), there exists $\alpha_1 \in \mathbb{R}$ such that $\sum_{k\in N} \alpha_{\{k\}}^c v(\{k\}) = \alpha_1 \sum_{k\in N} v(\{k\})$ for all $(N, v) \in \mathcal{G}_{c+}^N$ and all $c \in \mathbb{R}_+^N$ with $c_i \neq c_j$ for some $i, j \in N$. Indeed, for $(N, v), (N, w) \in \mathcal{G}_{nz+}^N$ defined in Step 2, taking $v(S) = 0$ for all $S \subseteq N$ with $|S| \geq 2$, from (4.23) we obtain

$$\sum_{k\in N} \alpha_{\{k\}}^c v(\{k\}) = \sum_{k\in N} \alpha_{\{k\}}^{c^*} w(\{k\}).$$

Denote $\alpha_1 = \frac{\sum_{k\in N} \alpha_{\{k\}}^{c^*} w(\{k\})}{\sum_{k\in N} w(\{k\})}$. Since $\sum_{k\in N} v(\{k\}) = \sum_{k\in N} w(\{k\})$, then

$$\sum_{k\in N} \alpha_{\{k\}}^c v(\{k\}) = \sum_{k\in N} \alpha_{\{k\}}^{c^*} w(\{k\}) = \alpha_1 \sum_{k\in N} w(\{k\}) = \alpha_1 \sum_{k\in N} v(\{k\}).$$

Similarly, applying (4.23) to (N, w) and (N, u) defined in Step 2, we obtain

$$\sum_{k\in N} \alpha_{\{k\}}^{c'} u(\{k\}) = \sum_{k\in N} \alpha_{\{k\}}^{c^*} w(\{k\}) = \alpha_1 \sum_{k\in N} u(\{k\}).$$

These two equations imply that the desired assertion holds if $\frac{v(\{i\})}{c_i} = 1$ and $\sum_{i\in N} c_i$ is a fixed real number. Let $\beta \in \mathbb{R}_+$ and $(N, v) \in \mathcal{G}_{\beta c+}^N = \mathcal{G}_{c+}^N$ be such that $\frac{v(\{i\})}{\beta c_i} = 1$, i.e. $\frac{v(\{i\})}{c_i} = \beta$. Since $(N, v) \in \mathcal{G}_{c+}^N$, from above it follows that $\mathcal{G}_{c+}^N = \mathcal{G}_{\beta c+}^N$, then $\sum_{k\in N} \alpha_{\{k\}}^c v(\{k\}) = \sum_{k\in N} \alpha_{\{k\}}^{\beta c} v(\{k\}) = \alpha_1 \sum_{k\in N} v(\{k\})$, as desired.

Step 4. The 'Step 2' and 'Step 3' considered the case that $c \in \mathbb{R}_+^N$ satisfies $c_i \neq c_j$ for some $i, j \in N$. Notice that if $c_k = c_h$ for all $k, h \in N$, each α_S^c in (4.22) does not have any bite.

Step 5. From Steps 2, 3 and 4, we can conclude that, taking $\alpha_S^c = \alpha_S^N$ for each $S \subseteq N$ with $|S| \geq 2$, and $\alpha_{\{k\}}^c = \alpha_1$ for all $k \in N$, (4.22) can be written as

$$\psi_i(N, v) = \frac{v(N)}{n} + \left[\alpha_1 \sum_{k\in N} v(\{k\}) + \sum_{S\subseteq N: |S|\geq 2} \alpha_S^N v(S)\right] \left(\frac{v(\{i\})}{K(v)} - \frac{1}{n}\right). \tag{4.24}$$

It is straightforward to show that $\alpha_S^N = \alpha_T^N$ for all $|S| = |T| \geq 2$ in (4.24) by anonymity. Therefore, setting $\alpha_{|S|}^{|N|} = \alpha_S^N$, we obtain

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{S \subseteq N} \alpha_{|S|}^{|N|} v(S) \left(\frac{v(\{i\})}{K(v)} - \frac{1}{n} \right),$$

as desired. □

Proof of Theorem 4.4

Before giving the technical proof of Theorem 4.4, we give a brief sketch. We first formulate what projection consistency requires from the relationship between $\alpha(N,v)$ and $\alpha(N\backslash\{j\}, v^x)$ (Step 1 below). Then, the key idea of the calculation of $\alpha(N,v)$ is to derive the coefficients α_S^N by considering some special games. We derive that (i) the coefficients corresponding to the grand coalition are equal for grand coalitions of the same size (Step 2), (ii) the coefficients corresponding to singletons are equal (Step 3), and (iii) the coefficients of other coalitions are zero (Step 3). After that, we derive the formula of $\alpha(N,v)$ (Step 4).

Proof of Theorem 4.4. Let $\alpha(N,v) = \sum\limits_{S \subseteq N} \alpha_S^N v(S)$ be a linear function with α_S^N being real numbers depending on S and N. We remark that, for any $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$, $x = \psi^\alpha(N,v)$ and $j \in N$, since $v^x(\{i\}) = v(\{i\})$ for all $i \in N \setminus \{j\}$, it holds that $(N\backslash\{j\}, v^x) \in \mathcal{G}_{nz}$. Clearly, $\psi_i^\alpha(N,v)$ can be rewritten as

$$\psi_i^\alpha(N,v) = \frac{v(N)}{n} + \alpha(N,v)\Big(\frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} - \frac{1}{n}\Big). \tag{4.25}$$

Step 1. Take any $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$ and $j \in N$. For $x = \psi^\alpha(N,v)$ and any $i \in N\backslash\{j\}$, we have

$$\begin{aligned}
\psi_i^\alpha(N\backslash\{j\}, v^x) =& \frac{v^x(N\backslash\{j\})}{n-1} + \alpha(N\backslash\{j\}, v^x)\Big(\frac{v^x(\{i\})}{\sum_{k \in N\backslash\{j\}} v^x(\{k\})} - \frac{1}{n-1}\Big) \\
=& \frac{v(N) - x_j}{n-1} + \alpha(N\backslash\{j\}, v^x)\frac{v(\{i\}) - \frac{1}{n-1}\sum_{k \in N\backslash\{j\}} v(\{k\})}{\sum_{k \in N\backslash\{j\}} v(\{k\})} \\
=& \frac{1}{n-1}\Big(v(N) - \frac{v(N)}{n} - \alpha(N,v)\frac{v(\{j\}) - \frac{1}{n}\sum_{k \in N} v(\{k\})}{\sum_{k \in N} v(\{k\})}\Big) \\
&+ \alpha(N\backslash\{j\}, v^x)\frac{v(\{i\}) - \frac{1}{n-1}\sum_{k \in N\backslash\{j\}} v(\{k\})}{\sum_{k \in N\backslash\{j\}} v(\{k\})} \\
=& \frac{v(N)}{n} - \alpha(N,v)\frac{v(\{j\}) - \frac{1}{n}\sum_{k \in N} v(\{k\})}{(n-1)\sum_{k \in N} v(\{k\})} \\
&+ \alpha(N\backslash\{j\}, v^x)\frac{v(\{i\}) - \frac{1}{n-1}\sum_{k \in N\backslash\{j\}} v(\{k\})}{\sum_{k \in N\backslash\{j\}} v(\{k\})},
\end{aligned}$$

where the third equality follows by substituting $x = \psi^\alpha(N,v)$.

From projection consistency, $\psi_i^\alpha(N,v) = \psi_i^\alpha(N\backslash\{j\},v^x)$ for all $i \in N\backslash\{j\}$, and thus we have

$$\begin{aligned}
&\frac{v(N)}{n} + \alpha(N,v)\frac{v(\{i\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})}{\sum_{k\in N} v(\{k\})} \\
&= \frac{v(N)}{n} - \alpha(N,v)\frac{v(\{j\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})}{(n-1)\sum_{k\in N} v(\{k\})} \\
&\quad + \alpha(N\backslash\{j\},v^x)\frac{v(\{i\}) - \frac{1}{n-1}\sum_{k\in N\backslash\{j\}} v(\{k\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{\alpha(N,v)}{\sum_{k\in N} v(\{k\})}\left(v(\{i\}) - \frac{\sum_{k\in N} v(\{k\})}{n} + \frac{v(\{j\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})}{n-1}\right) \\
&= \frac{\alpha(N\backslash\{j\},v^x)}{\sum_{k\in N\backslash\{j\}} v(\{k\})}\left(v(\{i\}) - \frac{\sum_{k\in N\backslash\{j\}} v(\{k\})}{n-1}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&-\frac{\sum_{k\in N} v(\{k\})}{n} + \frac{v(\{j\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})}{n-1} \\
&= \frac{-(n-1)\sum_{k\in N} v(\{k\}) + nv(\{j\}) - \sum_{k\in N} v(\{k\})}{n(n-1)} \\
&= \frac{-\sum_{k\in N} v(\{k\}) + v(\{j\})}{n-1},
\end{aligned}$$

it follows that

$$\frac{\alpha(N,v)}{\sum_{k\in N} v(\{k\})}\left(v(\{i\}) - \frac{\sum_{k\in N\backslash\{j\}} v(\{k\})}{n-1}\right) = \frac{\alpha(N\backslash\{j\},v^x)}{\sum_{k\in N\backslash\{j\}} v(\{k\})}\left(v(\{i\}) - \frac{\sum_{k\in N\backslash\{j\}} v(\{k\})}{n-1}\right).$$

Since there always exists a game $(N,v) \in \mathcal{G}_{nz}$ such that $v(\{i\}) - \frac{\sum_{k\in N\backslash\{j\}} v(\{k\})}{n-1} \neq 0$, then

$$\frac{\alpha(N,v)}{\sum_{k\in N} v(\{k\})} = \frac{\alpha(N\backslash\{j\},v^x)}{\sum_{k\in N\backslash\{j\}} v(\{k\})}$$

for this game.

With the notion $K = \sum_{k\in N} v(\{k\})$, we have

$$\frac{\alpha(N,v)}{K} = \frac{\alpha(N\backslash\{j\},v^x)}{K - v(\{j\})}. \tag{4.26}$$

By definition of α, v^x and $x = \psi^\alpha(N,v)$, we have

$$\begin{aligned}
&\alpha(N\backslash\{j\},v^x) \\
&= \sum_{S\subset N\backslash\{j\}} \alpha_S^{N\backslash\{j\}} v(S) + \alpha_{N\backslash\{j\}}^{N\backslash\{j\}}(v(N) - x_j)
\end{aligned}$$

$$= \sum_{S\subset N\backslash\{j\}} \alpha_S^{N\backslash\{j\}} v(S) + \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} \Big(\frac{n-1}{n} v(N) - \sum_{S\subseteq N} \alpha_S^N v(S) \frac{v(\{j\}) - \frac{K}{n}}{K}\Big). \tag{4.27}$$

Step 2. We show that $\alpha_{N\backslash\{i\}}^{N\backslash\{i\}} = \alpha_{N\backslash\{j\}}^{N\backslash\{j\}}$ for all $i, j \in N$. To show this, consider a game $(N, v) \in \mathcal{G}_{nz}$ with $i, j \in N$ such that $v(\{i\}) = v(\{j\})$ and $v(\{i\}) - \frac{\sum_{k\in N\backslash\{j\}} v(\{k\})}{n-1} \neq 0$ (It is possible since $|N| \geq 3$). From (4.26) we have $\alpha(N\backslash\{i\}, v^x) = \alpha(N\backslash\{j\}, v^x)$. That is, with (4.27),

$$\sum_{S\subset N\backslash\{i\}} \alpha_S^{N\backslash\{i\}} v(S) + \alpha_{N\backslash\{i\}}^{N\backslash\{i\}} \Big(\frac{n-1}{n} v(N) - \sum_{S\subseteq N} \alpha_S^N v(S) \frac{v(\{i\}) - \frac{K}{n}}{K}\Big)$$
$$= \sum_{S\subset N\backslash\{j\}} \alpha_S^{N\backslash\{j\}} v(S) + \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} \Big(\frac{n-1}{n} v(N) - \sum_{S\subseteq N} \alpha_S^N v(S) \frac{v(\{j\}) - \frac{K}{n}}{K}\Big). \tag{4.28}$$

The coefficients of the term $v(N)$ must be the same on both sides of (4.28), that is

$$\alpha_{N\backslash\{i\}}^{N\backslash\{i\}} \Big(\frac{n-1}{n} - \alpha_N^N \frac{v(\{i\}) - \frac{K}{n}}{K}\Big) = \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} \Big(\frac{n-1}{n} - \alpha_N^N \frac{v(\{j\}) - \frac{K}{n}}{K}\Big).$$

No matter what number α_N^N is, there is a game such that $\frac{n-1}{n} - \alpha_N^N \frac{v(\{i\}) - \frac{K}{n}}{K} \neq 0$. [6] Thus,

$$\alpha_{N\backslash\{i\}}^{N\backslash\{i\}} = \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} \quad \text{for all } i, j \in N. \tag{4.29}$$

Step 3. We derive $\alpha_T^{N\backslash\{k\}}$ for any $k \in N$ and $T \subset N\backslash\{k\}$, as follows. Using (4.29), it follows from (4.28) and the fact that $v(\{i\}) = v(\{j\})$, that

$$\sum_{S\subset N\backslash\{i\}} \alpha_S^{N\backslash\{i\}} v(S) = \sum_{S\subset N\backslash\{j\}} \alpha_S^{N\backslash\{j\}} v(S). \tag{4.30}$$

To derive $\alpha_T^{N\backslash\{k\}}$, we consider two cases with respect to $|T|$, $T \subset N$:

(i) For $T \subset N$ with $|T| \geq 2$, there exist $i, j \in N$ with $T \subset N\backslash\{i\}$ and $j \in T$. Clearly, the term $v(T)$ only appears in the left-hand side of (4.30), and thus it must be that $\alpha_T^{N\backslash\{i\}} = 0$. (We see this by taking two games that only differ in the worth of T.) Similarly, $\alpha_T^{N\backslash\{j\}} = 0$ for all $T \subset N\backslash\{j\}$ with $i \in T$ and $|T| \geq 2$.

Since $i, j \in N$ can be arbitrary two players, we obtain

$$\alpha_T^{N\backslash\{k\}} = 0, \tag{4.31}$$

for all $T \subset N$ with $|T| \geq 2$.

[6] For example, a game (N, v) such that there is an $i_0 \in N$ with $v(\{i_0\}) = 2$ and $v(\{j\}) = 1$ for all $j \in N \setminus \{i_0\}$ will do the job if $\alpha_N^N \neq (n+1)$, and a game (N, v) such that there are $i_0, i_1 \in N$ with $v(\{i_0\}) = v(\{i_1\}) = 2$ and $v(\{j\}) = 1$ for all $j \in N \setminus \{i_0, i_1\}$ will do the job otherwise.

(ii) Consider $T \subset N$ with $|T| = 1$. Since (4.30) holds under the condition $v(\{i\}) = v(\{j\})$, it must be that $\alpha_{\{j\}}^{N\backslash\{i\}} v(\{j\}) = \alpha_{\{i\}}^{N\backslash\{j\}} v(\{i\})$. (We see this by taking two games that only differ in the worths of $\{i\}$ and $\{j\}$ and using (4.31).) Since $v(\{i\}) = v(\{j\}) \neq 0$, then $\alpha_{\{j\}}^{N\backslash\{i\}} = \alpha_{\{i\}}^{N\backslash\{j\}}$. Similarly, it is clear that $\alpha_{\{k\}}^{N\backslash\{i\}} = \alpha_{\{k\}}^{N\backslash\{j\}}$ for all $k \in N\backslash\{i,j\}$.

Since $i, j \in N$ can be arbitrary two players, then $\alpha_{\{j\}}^{N\backslash\{i\}} = \alpha_{\{j\}}^{N\backslash\{k\}} = \alpha_{\{k\}}^{N\backslash\{j\}} = \alpha_{\{k\}}^{N\backslash\{i\}}$ for all $k \in N\backslash\{i,j\}$. Therefore,

$$\alpha_{\{i\}}^{N\backslash\{k\}} = \alpha_{\{j\}}^{N\backslash\{k\}}, \tag{4.32}$$

for all $i, j \in N\backslash\{k\}$ and all $k \in N$.

Step 4. Now, we can derive the desired assertion. Using (4.32), we denote $\beta^{n-1} = \alpha_i^{N\backslash\{k\}}$ for all $i \in N\backslash\{k\}$. Plugging (4.31) and (4.32) into the second line of (4.27), we have that for all $j \in N$,

$$\begin{aligned}\alpha(N\backslash\{j\}, v^x) &= \beta^{n-1} \sum_{k \in N\backslash\{j\}} v^x(\{k\}) + \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} v^x(N\backslash\{j\}) \\ &= \beta^{n-1}(K(v) - v(\{j\})) + \alpha_{N\backslash\{j\}}^{N\backslash\{j\}} v^x(N\backslash\{j\}).\end{aligned}$$

Consider a game $(N', v') \in \mathcal{G}_{nz}$ with $N' = N \cup \{j\}$, $j \notin N$, $(v')^x = v$. Similar as above, denoting $\beta^n = \alpha_i^N$ for all $i \in N$, we obtain

$$\alpha(N, v) = \beta^n K(v) + \alpha_N^N v(N). \tag{4.33}$$

Therefore, (4.26) can be written as

$$\begin{aligned}&\beta^n + \frac{\alpha_N^N v(N)}{K} \\ &= \beta^{n-1} + \frac{\alpha_{N\backslash\{j\}}^{N\backslash\{j\}} v^x(N\backslash\{j\})}{K - v(\{j\})} \\ &= \beta^{n-1} + \frac{\alpha_{N\backslash\{j\}}^{N\backslash\{j\}}}{K - v(\{j\})} \Big(\frac{n-1}{n} v(N) - \alpha_N^N v(N) \frac{v(\{j\}) - \frac{K}{n}}{K} - \beta^n (v(\{j\}) - \frac{K}{n}) \Big),\end{aligned}$$

where the second equality follows from (4.25).

Considering any game $(N, v) \in \mathcal{G}_{nz}$ such that $K(v) = n$ and $v(\{j\}) = 1$ (for example by taking a game $(N, v), |N| \geq 3$, such that $v(\{i_0\}) = 0.5$, $v(\{i_1\}) = 1.5$, and $v(\{k\}) = 1$ for all $k \in N\backslash\{i_0, i_1\}$), we obtain $\beta^n + \frac{\alpha_N^N v(N)}{n} = \beta^{n-1} + \frac{\alpha_{N\backslash\{j\}}^{N\backslash\{j\}} v(N)}{n}$. Since $v(N)$ can take any number, it must be that $\beta^{n-1} = \beta^n$ (by taking a game with $v(N) = 0$) and then $\alpha_N^N = \alpha_{N\backslash\{j\}}^{N\backslash\{j\}}$ (by taking a game with $v(N) > 0$). Hence, for any

$(N, v) \in \mathcal{G}_{nz}$, the above equation can be written as follows:

$$\frac{\alpha_N^N v(N)}{K} = \frac{\alpha_N^N}{K - v(\{j\})}\left(\frac{n-1}{n}v(N) - \alpha_N^N v(N)\frac{v(\{j\}) - \frac{K}{n}}{K} - \beta^n(v(\{j\}) - \frac{K}{n})\right),$$

which implies that

$$\big(K - v(\{j\})\big)\alpha_N^N v(N) = K\alpha_N^N\left(\frac{n-1}{n}v(N) - \alpha_N^N v(N)\frac{v(\{j\}) - \frac{K}{n}}{K} - \beta^n(v(\{j\}) - \frac{K}{n})\right). \tag{4.34}$$

There are two parameters α_N^N and β^n to be determined in (4.34). We distinguish the following two cases:

(i) If $\alpha_N^N = 0$, then β^n can take any real number, and thus (4.6) gives an affine combination of the ED value and ESD value.

(ii) If $\alpha_N^N \neq 0$, then

$$(K - v(\{j\}))v(N) = K\frac{n-1}{n}v(N) - (\alpha_N^N v(N) + K\beta^n)\big(v(\{j\}) - \frac{K}{n}\big). \tag{4.35}$$

It follows from (4.35) that

$$v(N)\big(v(\{j\}) - \frac{K}{n}\big) = (\alpha_N^N v(N) + K\beta^n)\big(v(\{j\}) - \frac{K}{n}\big).$$

Considering any game with $v(\{j\}) - \frac{K}{n} \neq 0$, we obtain

$$\alpha_N^N v(N) + K\beta^n = v(N). \tag{4.36}$$

Since this must hold for any $v(N)$, it follows that $\beta^n = 0$ (by taking $v(N) = 0$), and thus $\alpha_N^N = 1$ (by taking $v(N) > 0$). Hence, (4.6) gives the PD value.

We conclude from (4.33), and Cases (i) and (ii), that if ψ^α satisfies projection consistency, then $\alpha(N, v) = \beta \sum_{k \in N} v(\{k\})$, $\beta \in \mathbb{R}$, or $\alpha(N, v) = v(N)$. Therefore,

$$\psi^\alpha(N, v) = \begin{cases} \beta ESD(N, v) + (1-\beta)ED(N, v), & \text{if } \alpha(N, v) = \beta \sum_{k \in N} v(\{k\}), \\ PD(N, v), & \text{if } \alpha(N, v) = v(N). \end{cases}$$

The above assertion holds for $|N| \geq 3$. We now turn to the case $|N| = 2$. We already know that $\beta^{|N'|-1} = \beta^{|N'|}$ and $\alpha_{N'}^{N'} = \alpha_{N' \setminus \{j\}}^{N' \setminus \{j\}}$ for all $|N'| \geq 3$ and all $j \in N'$. Meanwhile, Cases (i) and (ii) hold for $|N'| \geq 3$. Specifically, taking $|N'| = 3$, we obtain that $\alpha(N, v) = v(N)$ or $\alpha(N, v) = \beta \sum_{k \in N} v(\{k\})$ for $|N| = |N'| - 1 = 2$, which immediately yields the desired assertion. □

Proof of Theorem 4.3. As mentioned in the main text, since the class of α-mollified values is contained in the class of α-mollified generalized values, and the resulting values in Theorem 4.4 are also members in the family of α-mollified values, we obtain Theorem 4.3 as a corollary of Theorem 4.4. A direct proof of Theorem 4.3 can be given by omitting Step 2 and Step 3 of the proof of Theorem 4.4. $\square$

Proof of Theorem 4.5. In view of (4.7), we have

$$\begin{aligned}
&\sum_{\pi\in\Pi(N)} \frac{1}{n!}\eta_i^{\pi,\alpha} \\
&= \sum_{\substack{\pi\in\Pi(N):\\ \pi(i)=1}} \frac{1}{n!}\left[v(\{i\}) + \sum_{j\in N:\pi(j)>\pi(i)} \frac{v(S_\pi^j)-v(S_\pi^j\setminus\{j\})-\frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{\pi(j)-1}\right] + \sum_{\substack{\pi\in\Pi(N):\\ \pi(i)=n}} \frac{1}{n!}\frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})} \\
&+ \sum_{\substack{\pi\in\Pi(N):\\ \pi(i)\neq 1,n}} \frac{1}{n!}\left[\frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})} + \sum_{j\in N:\pi(j)>\pi(i)} \frac{v(S_\pi^j)-v(S_\pi^j\setminus\{j\})-\frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{\pi(j)-1}\right] \\
&= \frac{v(\{i\})}{n} + \frac{n-1}{n}\frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})} + \frac{1}{n!}\sum_{\pi\in\Pi(N)}\sum_{j\in N:\pi(j)>\pi(i)} \frac{v(S_\pi^j)-v(S_\pi^j\setminus\{j\})-\frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{\pi(j)-1},
\end{aligned} \tag{4.37}$$

where the second equality holds since there are $(n-1)!$ (respectively $(n-1)(n-1)!$) permutations such that i has (respectively does not have) the first position, the same for the last position.

Note that for each coalition $S \subseteq N$ with $j \in S$, there are $(n-s)!(s-1)!$ permutations such that the first s players are exactly the members of S and j has the sth position. Hence, the third term of the right-hand side in (4.37) can be rewritten as follows:

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}\sum_{S\subseteq N:i,j\in S} \frac{v(S)-v(S\setminus\{j\})-\frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}}{s-1}\frac{(n-s)!(s-1)!}{n!} \\
&= \sum_{j\in N\setminus\{i\}}\sum_{S\subseteq N:i,j\in S} \frac{(n-s)!(s-2)!}{n!}[v(S)-v(S\setminus\{j\})] \\
&- \sum_{j\in N\setminus\{i\}}\left[\frac{v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}\sum_{S\subseteq N:i,j\in S}\frac{(n-s)!(s-2)!}{n!}\right].
\end{aligned} \tag{4.38}$$

We can rewrite the first term as

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}\sum_{S\subseteq N:i,j\in S} \frac{(n-s)!(s-2)!}{n!}[v(S)-v(S\setminus\{j\})] \\
&= \sum_{S\subseteq N:|S|\geq 2,i\in S}\sum_{j\in S\setminus\{i\}} \frac{(n-s)!(s-2)!}{n!}[v(S)-v(S\setminus\{j\})] \\
&= \sum_{s=2}^{n}\sum_{S\subseteq N:i\in S,|S|=s} \frac{(s-1)!(n-s)!}{n!}v(S) - \sum_{s=1}^{n-1}\sum_{S\subseteq N:i\in S,|S|=s} \frac{(s-1)!(n-s)!}{n!}v(S)
\end{aligned}$$

$$= \frac{(n-1)!(n-n)!}{n!}v(N) - \frac{(1-1)!(n-1)!}{n!}v(\{i\})$$
$$= \frac{v(N)}{n} - \frac{v(\{i\})}{n}.$$

Since

$$\sum_{S\subseteq N: i,j\in S} \frac{(n-s)!(s-2)!}{n!} = \sum_{s=2}^{n} \frac{(n-s)!(s-2)!}{n!} \frac{(n-2)!}{(n-s)!(s-2)!} = (n-1)\frac{(n-2)!}{n!} = \frac{1}{n},$$

we obtain from (4.37) and (4.38) that

$$\sum_{\pi\in\Pi(N)} \frac{1}{n!}\eta_i^{\pi,\alpha} = \frac{v(\{i\})}{n} + \frac{n-1}{n}\frac{v(\{i\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})} + \frac{v(N)}{n} - \frac{v(\{i\})}{n} - \frac{1}{n}\frac{\sum_{j\in N\setminus\{i\}} v(\{j\})\alpha(N,v)}{\sum_{k\in N} v(\{k\})}$$
$$= \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\alpha(N,v) + \frac{1}{n}\big(v(N) - \alpha(N,v)\big),$$

as desired. □

4.9 Conclusion

In this chapter, we have introduced a parametric family of values, called α-mollified values, that allocate the worth of the grand coalition based on proportional and equal division methods. We have axiomatized this family by employing efficiency, the balanced individual excess ratio property, continuity, weak additivity, anonymity, and no advantageous reallocation across individuals. Moreover, we have provided a novel analytical approach to select the PD value and the affine combinations of the ED and ESD values from the class of α-mollified values by imposing projection consistency. We did this using a larger class of α-mollified generalized values. Finally, we have implemented each member of this family based on a one-by-one formation of the grand coalition.

Proportional and equal division are two famous allocation principles in economics. Besides combining these two principles, the α-mollified values also allow that the part of the worth of the grand coalition to which we apply proportionality, respectively equality, depends on the worths of all coalitions. Whereas usual proportional and equal division solutions only take account of the worths of singletons and the grand coalition, the α-mollified values might thus depend on the worths of all coalitions.[7]

Since the family of α-mollified (generalized) values offers a simple yet flexible compromise between proportionality and equality principles, it is worthwhile to investigate some values within this family. In this framework, Chapter 5 will focus on a subfamily of α-mollified values.

[7] Other proportional solutions that take the worths of all coalitions into account are the proper Shapley values (Vorob'ev and Liapunov, 1998), the proportional Shapley value (Béal et al., 2018), and the proportional Harsanyi solution (Besner, 2020).

Future research on the α-mollified (generalized) values will be done on, for example, axiomatization and strategic implementation. In reality, the weight of a coalition may depend on the members of this coalition, and thus it is interesting to investigate what axiomatizations can be employed for the family of α-mollified generalized values. Besides, the dual value of an α-mollified value might be worth investigating. Some interesting values are the PANSC value (see Chapter 3), the EANSC value, and their convex combinations.

Chapter 5

Sharing the Surplus and Proportional Values

5.1 Introduction

Equal and proportional division are two basic principles in allocation problems. In TU-games, usually these principles are applied to a remainder of the surplus after each individual player is assigned an individual entitlement which can be equal to zero. For two-player games, this can be formalized in axioms such as *standardness* (assigning each player its stand-alone worth and allocating the surplus equally over all players), *egalitarian standardness* (ignoring individual entitlements and allocating the full worth equally over the players (which can be zero too)), and *proportional standardness* (allocating the full surplus proportional to the stand-alone worths of the players). For example, the Shapley value (Shapley, 1953a) and the equal surplus division value (Driessen and Funaki, 1991) satisfy standardness, the equal division value (axiomatized in van den Brink (2007)) satisfies egalitarian standardness, and various proportional values, such as the proportional value (Ortmann, 2000), the proportional Shapley value (Béal et al., 2018; Besner, 2019), the proper Shapley values (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015; van den Brink et al., 2020), and the proportional Harsanyi solution (Besner, 2020) satisfy proportional standardness. The values can be extended to games with more than two players by, for example reduced game consistency or balanced contributions type of axioms that relate payoffs of players in a game with their payoffs in a game on a reduced player set.

There is a large literature on 'equal sharing of the surplus' type of values. In contrast, values that appear to be 'proportional' are studied less, although proportional considerations play a central role in fair division problems as pointed out by a group of economists and academics, e.g., Brams and Taylor (1996), Chun (1988), Moulin (1987), Moulin (2004), Thomson (2015a), Tijs and Driessen (1986), and Young (1995). However, recently there is a growing literature on values that are based on proportionality, such as the values mentioned above.

In this chapter, which is based on Zou et al. (2020b), we provide a new family of values, called the *proportional surplus division values* which make a trade-off between a player's stand-alone worth and the average stand-alone worth, and allocate the remainder proportional to the stand-alone worths. Extreme cases of values in our family are the *proportional division value* based on Moriarity (1975) and Gangolly (1981), and shortly denoted by the PD value (see Chapter 2), and the *egalitarian proportional surplus division value*, shortly denoted by the EPSD value. The PD value allocates the worth of the grand coalition in proportion to players' stand-alone worth. The EPSD value is a new value that assigns to each player the average stand-alone worth, and then allocates the remainder of the worth of the grand coalition in proportion to players' stand-alone worth. The EPSD value focuses on egalitarianism in allocating the stand-alone worths by first assigning to every player the average of all stand-alone worths, whereas the PD value applies an egocentric principle and first assigns to each player its stand-alone worth. Both values apply proportionality in the allocation of the remaining surplus. Besides these two extreme values, our family consists of all convex combinations of the PD value and the EPSD value, and thus can be viewed as making a trade-off between egocentrism and egalitarianism. This family of values is in line with a recent and growing literature that combine different allocation principles by considering convex combinations of two extreme values, such as the egalitarian Shapley values (being convex combinations of the Shapley value and equal division value, see Joosten (1996) and van den Brink et al. (2013)), the consensus values (being convex combinations of the Shapley value and equal surplus division value, see Ju et al. (2007b)) and the family of convex combinations of the equal division value and the equal surplus division value (axiomatized in, e.g., van den Brink and Funaki (2009), van den Brink et al. (2016), Xu et al. (2015), and Ferrières (2017)). Also, our family of values is in line with a recent and growing literature on non-symmetric surplus sharing values, such as the weighted division value (Béal et al., 2015; Béal et al., 2016b), the weighted surplus division value (Calleja and Llerena, 2017; Calleja and Llerena, 2019), and the weighted equal allocation of non-separable contributions value (Hou et al., 2019) [1].

Besides several known axioms from the literature, we introduce new axioms concerning the separatorization of a player. Separatorization[2] refers to a player's obstruction of cooperation in the sense that the worth of any coalition containing him equals the sum of the stand-alone worths of the players in this coalition, while the worth of any coalition without this player remains unchanged. This is not to be confused with dummification as introduced in Béal et al. (2018) (strengthening nullification studied in Gómez-Rúa and Vidal-Puga (2010), Béal et al. (2016a), Ferrières (2017), Kongo (2018), Kongo (2019), and Kongo (2020)), where a player becomes a dummy player. The first axiom, called *proportional loss under separatorization*, requires

[1] The difference with our work and the papers here is that our weights are endogenous and they have exogenously given weights.

[2] We thank André Casajus for suggesting the names of separator and separatorization at the 15th European Meeting on Game Theory.

that if a player becomes a separator, then all other player's payoff change in proportion to their stand-alone worths. The second axiom, called *proportional balanced contributions under separatorization*, requires that, for any two players, the effects of one of them becoming a separator on the payoff of the other, are proportional to their stand-alone worths.

Following Ferrières (2017) and Béal et al. (2018), we identify the consequence of imposing either of the aforementioned axioms in addition to the classical axiom of efficiency. It turns out that the resulting values have all in common that they split the worth of the grand coalition in proportion to players' stand-alone worth. Moreover, any member of this family is uniquely determined by a value defined on additive games (being games where all players are separators and thus the worth of every coalition equals the sum of the stand-alone worths of the players in that coalition). Subsequently, we characterize a family of values for quasi-additive games by means of known axioms of efficiency, anonymity, a weak version of weak no advantageous reallocation, and continuity, which generalizes a remarkable result for rights problems in Chun (1988). By combining the axioms in these results and using weak linearity instead of continuity, the family of affine combinations of the PD and EPSD values is characterized. Replacing anonymity with superadditive monotonicity (Ferrières, 2017) and weak desirability, we derive an axiomatization of the family of convex combinations of the PD and EPSD values. Besides, we show how specific values are singled out by using a parametrized axiom which puts a certain lower bound on the payoffs of individual players.

This chapter is organized as follows. Section 5.2 provides some notation and definitions. Section 5.3 introduces the concept of proportional surplus division values. Section 5.4 and Section 5.5 contain the results. Section 5.6 shows the logical independence of the axioms in the characterization results. All proofs are provided in Section 5.7. Section 5.8 presents a conclusion.

5.2 Definitions and Notation

We recall some definitions from Chapter 1 that are used in this chapter. Recall that $\mathcal{G}_{nz}^N$ denotes the class that consists of all individually positive and individually negative games on a specific player set N. $\mathcal{A}_{nz}^N$ denotes the class of additive games from $\mathcal{G}_{nz}^N$, and $\mathcal{QA}_{nz}^N$ denotes the class of quasi-additive games from $\mathcal{G}_{nz}^N$. Béal et al. (2018) provide many applications of the restricted class of games $\mathcal{G}_{nz}^N$, such as land production economies, telecommunication problems, and sequencing/queueing problems. We restrict our discussion to this class of games. Let $\mathcal{G}_{nz}^{\geq 3} = \{(N,v) \in \mathcal{G}_{nz}^N \mid |N| \geq 3\}$, $\mathcal{QA}_{nz}^{\geq 3} = \{(N,v) \in \mathcal{QA}_{nz}^N \mid |N| \geq 3\}$, and $\mathcal{A}_{nz}^{\geq 3} = \{(N,v) \in \mathcal{A}_{nz}^N \mid |N| \geq 3\}$.

As mentioned in the introduction, this chapter will focus on a new class of values that is closely related to the proportional division value. The definition of the proportional division (PD) value is given as follows.

The *PD value* on $\mathcal{C} \subseteq \mathcal{G}^N_{nz}$ is defined for all $(N, v) \in \mathcal{C}$ and $i \in N$ by

$$PD_i(N, v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N).$$

We present three known properties of values on special classes of TU-games. Note that the second property is not yet discussed in Chapter 1.

- **Weak desirability**. For all $(N, v) \in \mathcal{A}^N_{nz}$ and $i, j \in N$ such that $v(\{i\}) \geq v(\{j\})$[3], it holds that $\psi_i(N, v) \geq \psi_j(N, v)$.
- **Weak no advantageous reallocation**. For all quasi-additive games $(N, v), (N, w) \in \mathcal{QA}^N_{nz}$ and $T \subseteq N$ such that $\sum_{i\in T} v(\{i\}) = \sum_{i\in T} w(\{i\})$, $v(\{i\}) = w(\{i\})$ for all $i \in N\backslash T$, and $v(N) = w(N)$, it holds that $\sum_{i\in T} \psi_i(N, v) = \sum_{i\in T} \psi_i(N, w)$.
- **Continuity**. For all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{QA}^N_{nz}$ such that $\lim\limits_{k\to\infty}(N, w_k) = (N, v)$, it holds that $\lim\limits_{k\to\infty} \psi(N, w_k) = \psi(N, v)$.

Weak desirability states that if i's contributions are greater than or equal to j's contributions in an additive game, then i should receive at least j's payoff. The condition $v(S \cup \{i\}) \geq v(S \cup \{j\})$ in weak desirability is equivalent to $v(\{i\}) \geq v(\{j\})$ for additive games.

Weak no advantageous reallocation states that transfers of individual productivities across a subset of players do not affect the total payoffs of this coalition.

Continuity states that a small change in the parameters of the game causes only a small change in the payoff.

Weak no advantageous reallocation considers two games that the worths of intermediate coalitions are determined by the stand-alone worths, whereas weak no advantageous reallocation across individuals in Chapter 4 considers the fixed worths of intermediate coalitions of two games. This axiom and continuity are required only for quasi-additive games. Moulin (1987) and Chun (1988), respectively, used the last two axioms in surplus problems and rights problems, which can be considered as quasi-additive games.

5.3 Proportional surplus division values

In this chapter, we characterize families of combinations of the PD value and a new value called EPSD value. We begin this section by defining this new value.

The *egalitarian proportional surplus division (EPSD) value* on $\mathcal{C} \subseteq \mathcal{G}^N_{nz}$ is defined for all $(N, v) \in \mathcal{C}$ and $i \in N$ by

$$EPSD_i(N, v) = \frac{1}{|N|} \sum_{j\in N} v(\{j\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} [v(N) - \sum_{j\in N} v(\{j\})].$$

[3] Notice that for additive games this is equivalent to $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

Similar as other values mentioned before, the EPSD value is based on the idea of first assigning individual entitlements to the players, and then allocating the remainder of $v(N)$ over all players using an egalitarian or proportionality principle. In the case of the EPSD value, we first assign to every player the average stand-alone worth, and then allocate the remainder proportional to the stand-alone worths. Thus, the individual entitlements reflect egalitarianism in the sense that all stand-alone worths are equally shared among all players. However, discrimination is made in the allocation of the remainder which is allocated proportional to the stand-alone worths.

To compare the EPSD value with the PD value, notice that the PD value can be written as

$$PD_i(N,v) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}(v(N) - \sum_{j\in N} v(\{j\}))$$

for all $(N,v) \in \mathcal{C}$ and $i \in N$. So, splitting the allocation of $v(N)$ into (i) the allocation of all stand-alone worths, and (ii) the allocation of the *remainder* of the worth of the grand coalition after subtracting all stand-alone worths, both the PD and EPSD values allocate the remainder proportional to the stand-alone worths, but the PD value also discriminates in the allocation of the stand-alone worths (in the sense that every players gets its own stand-alone worth), whereas the EPSD value only discriminates with respect to allocating the remainder (allocating it proportional to the stand-alone worths) and allocates all stand-alone worths equally over all players. Although our main motivation for the EPSD value comes from the axiomatizations in Section 5.4, a direct motivation for the EPSD value is that, similar as in other resource allocation models, such as bankruptcy problems, applying proportional division might leave some players (with relatively small stand-alone worths) with a (too) small share in the resource. This can be 'solved' by allocating an initial uniform share to all players. In the EPSD value this share is restricted by the sum of all stand-alone worths.

Table 5.1 clarifies the difference among the ED, ESD, PD and EPSD values by the way they allocate (i) the sum of all stand-alone worths $\sum_{j\in N} v(\{j\})$, and (ii) the surplus $v(N) - \sum_{j\in N} v(\{j\})$ that is left from the worth of the grand coalition. These are allocated either equally over the players (E-principle) or proportional to their stand-alone worths (P-principle). Whereas the ED, ESD and PD values have been studied in the literature, the EPSD value is not and thus fills a gap.

TABLE 5.1: Values and division principles

Values	$\sum_{j\in N} v(\{j\})$		$v(N) - \sum_{j\in N} v(\{j\})$	
	E-principle	P-principle	E-principle	P-principle
ED	$\surd$		$\surd$	
ESD		$\surd$	$\surd$	
PD		$\surd$		$\surd$
EPSD	$\surd$			$\surd$

Obviously, whereas the ED, respectively PD, values are the E-principle, respectively P-principle, in both aspects, the ESD value and the EPSD value reflect equal as well as proportional sharing. Specifically, the ESD value allocates the amounts of $\sum_{j\in N} v(\{j\})$ and $v(N) - \sum_{j\in N} v(\{j\})$ by respectively applying the P-principle and the E-principle, while the EPSD value does it the other way around.

Example 5.1. Consider a game (N, v), where $N = \{1, 2, 3\}$ and the characteristic function is given as $v(\emptyset) = 0, v(\{1\}) = v(\{2\}) = 1, v(\{3\}) = 2, v(\{1,2\}) = 4, v(\{1,3\}) = 6, v(\{2,3\}) = 8, v(\{1,2,3\}) = 9$. We compute the ED, ESD, PD, and EPSD values as

$$ED = (3, 3, 3); \quad ESD(N, v) = (\frac{8}{3}, \frac{8}{3}, \frac{11}{3});$$
$$PD = (\tfrac{9}{4}, \tfrac{9}{4}, \tfrac{9}{2}); \quad EPSD(N, v) = (\frac{31}{12}, \frac{31}{12}, \frac{23}{6}).$$

All the four values are on one line. The ED and PD values are two extreme points and the EPSD and ESD values are on the line segment.

In this chapter, we consider combinations of the EPSD value and the PD value. For all $(N, v) \in \mathcal{G}^N_{nz}$ and $\alpha \in \mathbb{R}$, the corresponding value φ^α, called *α-proportional surplus division value*, is defined by

$$\varphi^\alpha(N, v) = \alpha EPSD(N, v) + (1 - \alpha) PD(N, v).$$

It is straightforward to verify that for all $(N, v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, it holds that

$$\varphi^\alpha_i(N, v) = \frac{\alpha}{n} \sum_{j\in N} v(\{j\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} \alpha v(\{j\})]. \tag{5.1}$$

The value $\varphi^\alpha(N, v)$ first assigns to every player the fraction α of the average stand-alone worth, and then allocates the remainder (which might be negative) proportional to the stand-alone worths.

Alternatively, (5.1) can be rewritten as follows.

$$\varphi^\alpha_i(N, v) = \frac{\alpha}{n} \sum_{j\in N} v(\{j\}) + (1 - \alpha) v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} v(\{j\})]. \tag{5.2}$$

This formulation makes clear that an α-proportional surplus division value can also be interpreted as first assigning to every player affine combination of the average and its own stand-alone worth, and then allocating the surplus $v(N) - \sum_{j\in N} v(\{j\})$ proportional to the stand-alone worths. The payoff $\frac{1}{n} \sum_{j\in N} v(\{j\})$ can be viewed as an egalitarian distribution, while the payoff $v(\{i\})$ can be interpreted as an egocentric distribution of the stand-alone worths. Hence, if $\alpha \in [0, 1]$ the value $\varphi^\alpha(N, v)$ can be seen as making a trade-off between egocentrism and egalitarianism, where the coefficient $\alpha \in [0, 1]$ is a measure of the social preference between egocentrism and egalitarianism. In the extreme cases, $\alpha = 1$ yields the EPSD value and reflects that the society prefers egalitarianism, while $\alpha = 0$ yields the PD value and reflects that the society prefers egocentrism.

In what follows, we refer to the class of values as 'proportional surplus division values'.

5.4 Axiomatizations of the family of proportional surplus division values

In this section, we provide axiomatizations of the family of proportional surplus division values using known axioms and either one of two new axioms. These new axioms are concerned with how a value should respond to the separatorization of a player in a game. Separatorization of a player refers to the complete loss of productive potential of cooperation within any coalition containing this player. Notice that, in case the worth of a coalition is less than the sum of the stand-alone worths of the players in the coalition, then separatorization results in a higher worth of the coalition since the mutual destruction aspect that is in the game is removed.

More specifically, a new game is constructed from the original one, in which the worth of any coalition containing the separator is equal to the sum of the stand-alone worths of the players in this coalition. A separatorization can also lead from a complete disaster (total mutual obstruction) of a coalition to a worth that corresponds to the sum of the singleton worths. Different than a nullifying player, whose entrance to a coalition implies that the total worth becomes zero, the entrance of a separator still allows the players to earn their stand-alone worth. This might occur, for example in a peaceful bargaining situation between countries, where a separator is a kind of 'saboteur' who makes negotiations and cooperation to fail, but all countries can still 'produce' in their own country. On the other hand, in peace negotiations in a situation of war, it can be that there is a nullifying player whose entrance implies failure of the peace negotiations resulting in destructive warfare.

Formally, for $(N, v) \in \mathcal{G}^N_{nz}$ and $h \in N$, we denote by (N, v^h) the game in which player h becomes a separator: For every $S \subseteq N$,

$$v^h(S) = \begin{cases} \sum_{j \in S} v(\{j\}) & \text{if } h \in S, \\ v(S) & \text{otherwise.} \end{cases}$$

For $S = \{i_1, i_2, \ldots, i_s\} \subseteq N$, consider the sequence $(v^{i_1}, (v^{i_1})^{i_2}, \ldots, (((v^{i_1})^{i_2})^{\cdots i_{s-1}})^{i_s})$. Note that $(v^i)^j = (v^j)^i$ for every pair $i, j \in N$, so that (N, v^S), with $v^S = (((v^{i_1})^{i_2})^{\cdots i_{s-1}})^{i_s}$ in the sequence above, is well-defined for every coalition $S \subseteq N$, and does not depend on the order in which the players become separators. Specifically, (N, v^N) is the corresponding additive game of (N, v) such that $v^N(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$.

There exist several axioms which evaluate the consequences of such operation that a player takes as special role in TU-games; we refer to balanced contributions in Myerson (1980), the veto equal loss property in van den Brink and Funaki (2009), the nullified equal loss property in Ferrières (2017), Kongo (2018), and Kongo (2020), and proportional balanced contributions under dummification in Béal et al. (2018). Similarly, we will introduce two new axioms concerning the separatorization.

5.4.1 Proportional loss under separatorization

The first new axiom is *proportional loss under separatorization*, which states that, if a player becomes a separator, then any two other players are affected proportionally to their stand-alone worths. Obviously, this axiom is considered only for games with at least three players.

- **Proportional loss under separatorization**. For all $(N, v) \in \mathcal{G}_{nz}^{\geq 3}$, all $h \in N$, and all $i, j \in N \backslash \{h\}$, it holds that

$$\frac{\psi_i(N, v) - \psi_i(N, v^h)}{v(\{i\})} = \frac{\psi_j(N, v) - \psi_j(N, v^h)}{v(\{j\})}.$$

Note that $(N, v^h) \in \mathcal{G}_{nz}^{\geq 3}$ for all $(N, v) \in \mathcal{G}_{nz}^{\geq 3}$ and $h \in N$, since the stand-alone worths do not change when a player becomes a separator. We begin the axiomatic study by uncovering two useful properties implied by the combination of efficiency and proportional loss under separatorization.

The first property says that under these two axioms, if two values coincide on the class of additive games, then they coincide on the class of all games in $\mathcal{G}_{nz}^{\geq 3}$.

Lemma 5.1. *Consider two values ψ and φ satisfying efficiency and proportional loss under separatorization on $\mathcal{G}_{nz}^{\geq 3}$ such that $\psi = \varphi$ on $\mathcal{A}_{nz}^{\geq 3}$. Then $\psi = \varphi$ on $\mathcal{G}_{nz}^{\geq 3}$.*

The proof of this lemma and of all other results in this chapter can be found in Section 5.7.

The second property follows from Lemma 5.1 and describes a relation between the payoffs of any game in $\mathcal{G}_{nz}^{N}$ and the game where all players become separators.

Lemma 5.2. *If a value ψ on $\mathcal{G}_{nz}^{\geq 3}$ satisfies efficiency and proportional loss under separatorization, then*

$$\psi_i(N, v) - \psi_i(N, v^N) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) - v(\{i\}) \tag{5.3}$$

for all $(N, v) \in \mathcal{G}_{nz}^{\geq 3}$ and $i \in N$.

Remark 5.1. The term $\frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$ in (5.3) causes that any value satisfying efficiency and proportional loss under separatorization is not linear. As a consequence, there is no value on $\mathcal{G}_{nz}^{\geq 3}$ that satisfies efficiency, proportional loss under separatorization, and linearity.

Remark 5.2. The converse of Lemma 5.2 does not hold since a value with the form of (5.3) need not satisfy efficiency as can be illustrated by the value $\varphi = PD + a$, where $a \in \mathbb{R}^N$ is such that $\sum_{i \in N} a_i \neq 0$, which also satisfies (5.3) but not efficiency. However, every value of the form given in (5.3) satisfies proportional loss under separatorization, which follows since applying (5.3) to (N, v^h), $h \in N$, and using the fact that $v^h(N) = \sum_{j \in N} v(\{j\})$, we have $\psi_i(N, v^h) - \psi_i(N, v^N) = 0$ for $i \in N$. Subtracting this equality from (5.3) yields $\psi_i(N, v) - \psi_i(N, v^h) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) - v(\{i\})$, as desired.

The following theorem characterizes a family of values on a restrictive domain of quasi-additive games.

Theorem 5.1. *A value ψ on $\mathcal{QA}_{nz}^{\geq 3}$ satisfies efficiency, anonymity, weak no advantageous reallocation, and continuity if and only if there exists a continuous function $g : \mathbb{R} \backslash \{0\} \times \mathbb{R} \to \mathbb{R}$ such that*

$$\psi_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) - \left(\frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} - \frac{1}{n} \right) g\Big(\sum_{j \in N} v(\{j\}), v(N)\Big) \tag{5.4}$$

for all $(N, v) \in \mathcal{QA}_{nz}^{\geq 3}$ and $i \in N$.

Remark 5.3. Chun (1988) shows a similar result (Theorem 1) for the situation that the sum of all stand-alone worths is nonzero. If continuity in Theorem 5.1 is replaced by the weaker condition that ψ is continuous at least at one point of its domain, then it affects only the properties of g which is no longer required to be continuous, but does not affect (5.4); we refer to Remark 1 in Chun (1988).

The ED, ESD, PD, and EPSD values restricted to the subclass of quasi-additive games are members of the family characterized by Theorem 5.1. They are obtained by setting $g(\sum_{j \in N} v(\{j\}), v(N))$ equal to $v(N)$, $v(N) - \sum_{j \in N} v(\{j\})$, 0, and $\sum_{j \in N} v(\{j\})$, respectively.

Among the values characterized in Theorem 5.1, only the affine combinations of the PD and EPSD values satisfy proportional loss under separatorization and weak linearity. This result still holds even if the domain $\mathcal{QA}_{nz}^{\geq 3}$ is extended to the domain $\mathcal{G}_{nz}^{\geq 3}$.

Theorem 5.2. *A value ψ on $\mathcal{G}_{nz}^{\geq 3}$ satisfies efficiency, anonymity, weak no advantageous reallocation, proportional loss under separatorization, and weak linearity if and only if there is $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1 - \alpha) PD$.*

A subfamily of affine combinations of the PD value and the EPSD value on $\mathcal{G}_{nz}^{N}$ is characterized by imposing superadditive monotonicity (see Page 16) on $\mathcal{G}_{nz}^{\geq 3}$.

Theorem 5.3. *A value ψ on $\mathcal{G}_{nz}^{\geq 3}$ satisfies efficiency, anonymity, weak no advantageous reallocation, proportional loss under separatorization, weak linearity, and superadditive monotonicity if and only if there is $\alpha \in [0, \frac{n}{n-1}]$ such that $\psi = \alpha EPSD + (1 - \alpha) PD$.*

Remark 5.4. Define the following modification of the $EPSD$ value:

$$EPSD'_i(N,v) = \frac{1}{n-1}\sum_{j\in N\setminus\{i\}} v(\{j\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} v(\{j\})].$$

The difference between the $EPSD$ value and $EPSD'$ is that in the latter one, each player $i \in N$ first gets the average stand-alone worth over all other players $j \in N\setminus\{i\}$ instead of the average stand-alone worth over all players as in the EPSD value. Then, the family of values characterized in Theorem 5.3 can also be expressed as $\{\alpha' EPSD' + (1-\alpha')PD \mid \alpha' \in [0,1]\}$ with $\alpha' = \frac{n-1}{n}\alpha$.

We identify the family of convex combinations of the PD value and the EPSD value on $\mathcal{G}^N_{nz}$ by using weak desirability instead of anonymity in Theorem 5.3.

Theorem 5.4. *A value ψ on $\mathcal{G}^{\geq 3}_{nz}$ satisfies efficiency, weak no advantageous reallocation, proportional loss under separatorization, weak linearity, superadditive monotonicity, and weak desirability if and only if there is $\alpha \in [0,1]$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

The proof uses the following lemma, which reveals that weak desirability together with some of the axioms in Theorem 5.3 imply anonymity.

Lemma 5.3. *On $\mathcal{G}^{\geq 3}_{nz}$, efficiency, weak no advantageous reallocation, proportional loss under separatorization, and weak desirability imply anonymity.*

5.4.2 Proportional balanced contributions under separatorization

Notice that in the results of Section 5.4.1, we had to exclude two-player games. The reason is that proportional loss under separatorization compares the effect on the payoffs of two distinct players by separatorization of yet another (third) player, and thus involves three players. In contrast, we introduce *proportional balanced contributions under separatorization* which states that any two players are affected proportionally to their stand-alone worths if the other becomes a separator.

- **Proportional balanced contributions under separatorization**. For all $(N,v) \in \mathcal{G}^N_{nz}$ and all $i,j \in N$, it holds that

$$\frac{\psi_i(N,v) - \psi_i(N,v^j)}{v(\{i\})} = \frac{\psi_j(N,v) - \psi_j(N,v^i)}{v(\{j\})}.$$

Since proportional balanced contributions under separatorization only compares the effect on the payoffs of two players by mutually becoming a separator, and thus involves only two players, it turns out that using this axiom instead of proportional loss under separatorization, Lemma 5.1 and Lemma 5.2 can be stated also for two-player games.

Lemma 5.4. *Consider two values ψ and φ satisfying efficiency and proportional balanced contributions under separatorization on $\mathcal{G}^N_{nz}$ such that $\psi = \varphi$ on $\mathcal{A}^N_{nz}$. Then $\psi = \varphi$ on $\mathcal{G}^N_{nz}$.*

Lemma 5.5. *If a value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency and proportional balanced contributions under separatorization, then*

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\})$$

for all $(N,v) \in \mathcal{G}_{nz}^N$ and $i \in N$.

Comparing Lemma 5.2 and Lemma 5.5, efficiency together with either proportional loss under separatorization or proportional balanced contributions under separatorization generate the family of values with the same formula. Thus, we may adopt proportional balanced contributions under separatorization instead of proportional loss under separatorization for Theorems 5.2, 5.3, and 5.4 given in Section 5.4.1.

Theorem 5.5. *Let ψ be a value on $\mathcal{G}_{nz}^{\geq 3}$ that satisfies efficiency, weak no advantageous reallocation, proportional balanced contributions under separatorization, and weak linearity. Then,*

(i) *ψ satisfies anonymity if and only if there is $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

(ii) *ψ satisfies anonymity and superadditive monotonicity if and only if there is $\alpha \in [0, \frac{n}{n-1}]$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

(iii) *ψ satisfies weak desirability and superadditive monotonicity if and only if there is $\alpha \in [0,1]$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

Remark 5.5. Although Lemmas 5.4 and 5.5 are valid also for two-player games, in Theorem 5.5, the restriction $|N| \neq 2$ cannot be omitted. Specifically, if $|N| = 2$ then, for example, the value defined by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\}) + \frac{(v(\{i\}))^2}{\sum_{j\in N}(v(\{j\}))^2} \sum_{j\in N} v(\{j\}) \tag{5.5}$$

satisfies all axioms, but it does not coincide with $\alpha EPSD + (1-\alpha)PD$ for any $\alpha \in \mathbb{R}$.

Remark 5.6. In Theorems 5.2–5.5, weak no advantageous reallocation can be replaced by the following stronger axiom.

- **Transfer rationality**. For any additive games $(N,v), (N,w) \in \mathcal{A}_{nz}^N$ such that $\sum_{j\in N} v(\{j\}) = \sum_{j\in N} w(\{j\})$, it holds that $\psi_i(N,v) - \psi_i(N,w) = \beta[v(\{i\}) - w(\{i\})]$ for some $\beta \in \mathbb{R}$ and all $i \in N$.

Transfer rationality states that an additive game is constructed from the initial additive game by transfering individual productivities across the players, then the difference in payoffs for any two players is proportional to the difference in their standalone worths. In this way, the restriction $|N| \neq 2$ can be taken out in Theorem 5.5.

5.5 Axiomatizations of the α-proportional surplus division value

We now provide characterizations for specific values from the family of proportional surplus division values. For this, we use a parametrized axiom, depending on fixed $\alpha \in [0,1]$, such that it singles out the corresponding value φ^α from the class $\{\alpha EPSD + (1-\alpha)PD \mid \alpha \in [0,1]\}$.

The *α-egalitarian inessential game property* makes a trade-off between egalitarianism (i.e. every palyer gains $\frac{v(N)}{n}$) and egocentrism (i.e. player i gains her own stand-alone worth $v(\{i\})$) in additive games, by requiring that in such games a fraction α of the worth of the grand coalition is allocated equally over the players, and the players additionally keep the complementary fraction $(1-\alpha)$ of their own stand-alone worth.

- **α-egalitarian inessential game property**. Let $\alpha \in [0,1]$. For every additive game $(N,v) \in \mathcal{A}^N_{nz}$ and all $i \in N$, it holds that $\psi_i(N,v) = (1-\alpha)v(\{i\}) + \alpha\frac{v(N)}{n}$.

When $\alpha = 0$ this yields the well-known inessential game property, while $\alpha = 1$ yields equal division for inessential games as introduced in Ferrières (2017). Further, a higher (respectively lower) α reflects a more egalitarian (respectively egocentric) society. Adding this axiom to the axioms of efficiency and proportional loss under separatorization characterizes the corresponding proportional surplus division value (except for two-player games).

Theorem 5.6. *Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency, proportional loss under separatorization, and the α-egalitarian inessential game property if and only if $\psi = \varphi^\alpha$.*

Next, we provide an alternative characterization of a specific proportional surplus division value using another parameterized axiom. For $\alpha \in [0,1]$, we call a game *α-essential* if $\sum_{i\in N} \alpha v(\{i\}) \leq v(N)$. Clearly, for $\alpha = 0$ this boils down to $v(N) \geq 0$, while for $\alpha = 1$ this is weak essentiality. The following axiom imposes a lower bound on the payoffs of players in α-essential games between zero and the average stand-alone worth. Specifically, it requires that each player receives at least a fixed fraction $\alpha \in [0,1]$ of the average stand-alone worth if it is feasible to do so.

- **α-reasonable lower bound**. Let $\alpha \in [0,1]$. For every α-essential game $(N,v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, $\psi_i(N,v) \geq \frac{\alpha}{n}\sum_{j\in N} v(\{j\})$.

We compare this axiom with a known lower bound axiom for α-essential games which requires that in such games, every player earns at least a fraction α of its stand-alone worth, see van den Brink et al. (2016).[4]

[4] Recall that α-individual rationality is used to characterize the convex combinations of the ED and ESD values in van den Brink et al. (2016) and Xu et al. (2015).

- **α-individual rationality**. Let $\alpha \in [0,1]$. For every α-essential game $(N,v) \in \mathcal{G}_{nz}^N$ and all $i \in N$, $\psi_i(N,v) \geq \alpha v(\{i\})$.

Notice that α-individual rationality relies on egocentrism and α-reasonable lower bound rests on egalitarianism. In both cases, α can be seen as a social selfish coefficient balancing the preference between egalitarianism and egocentrism. For $\alpha = 0$ both boil down to *nonnegativity*, requiring that $\psi_i(N,v) \geq 0$ for all $i \in N$ and every game $(N,v) \in \mathcal{G}_{nz}^N$ with $v(N) \geq 0$. For $\alpha = 1$, 1-individual rationality is the usual *individual rationality* axiom requiring that in a weakly essential game every player earns at least its stand-alone worth, while 1-reasonable lower bound guarantees every player at least the average stand-alone worth. It turns out that adding α-reasonable lower bound to efficiency and proportional loss under separatorization characterizes the corresponding φ^α, while adding α-individual rationality yields *only* the PD value.

Theorem 5.7. *Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, proportional loss under separatorization, and α-reasonable lower bound if and only if $\psi = \varphi^\alpha$.*

Corollary 5.1. Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, proportional loss under separatorization, and α-individual rationality if and only if $\psi = PD$.

Remark 5.7. In the proof of Theorem 5.7, α-reasonable lower bound is only used to determine a payoff vector of every game $(N,v) \in \mathcal{G}_{nz}^N$ in which $\sum_{i\in N} \alpha v(\{i\}) = v(N)$. Thus, it can be replaced with other similar axioms suitable for this task, such as the *Equal treatment for α-dummifying player* axiom which requires that an α-dummifying player (Xu et al., 2015) earns at least an equal share in the worth of the grand coalition.

- **Equal treatment for α-dummifying player**. Let $\alpha \in [0,1]$. For all $(N,v) \in \mathcal{G}_{nz}^N$, it holds that $\psi_i(N,v) = \frac{v(N)}{n}$, where player $i \in N$ is an *α-dummifying player* in (N,v): $v(S) = \alpha \sum_{j\in S} v(\{j\})$ for all $S \subseteq N$ such that $i \in S$ and $|S| \geq 2$.

Theorems 5.6 and 5.7 immediately imply axiomatic characterizations of the PD value and the EPSD value by taking φ^α with $\alpha = 0, 1$, respectively.

All characterization results in this subsection still hold by replacing proportional loss under separatorization with proportional balanced contributions under separatorization. In this way, the restriction $|N| \neq 2$ can be taken out.

Remark 5.8. The difference between the PD value and the proportional Shapley value is pinpointed to one axiom. With Theorem 5.6, it immediately follows that the PD value is characterized by efficiency, proportional balanced contributions under separatorization, and the inessential game property. Notice that as Theorem 1.14 shown, Béal et al. (2018) offer a characterization of the proportional Shapley value on $\mathcal{G}_{nz}^N$ by employing efficiency, proportional balanced contributions under dummification, and the inessential game property.

5.6 Independence of axioms

Logical independence of the axioms used in the characterization results can be shown by the following alternative values.

Theorem 5.2:

(i) The value $\psi(N,v) = \mathbf{0}$ for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ satisfies all axioms except efficiency.

(ii) The value on $\mathcal{G}_{nz}^{\geq 3}$ defined for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ with $N = \{1,2,...,n\}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\}) + \frac{i}{\sum_{j\in N} j} \sum_{j\in N} v(\{j\}) \tag{5.6}$$

satisfies all axioms except anonymity.

(iii) The value defined by (5.5) satisfies all axioms except weak no advantageous reallocation.

(iv) The ED value satisfies all axioms except proportional loss under separatorization.

(v) The value on $\mathcal{G}_{nz}^{\geq 3}$ defined for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - \left(v(\{i\}) - \frac{1}{n} \sum_{j\in N} v(\{j\}) \right) \left(\frac{1}{2} \right)^{\sum\limits_{j\in N} v(\{j\})}$$

satisfies all axioms except weak linearity.

Theorem 5.3 and Theorem 5.4:

(i) The value $\psi(N,v) = \mathbf{0}$ for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ satisfies all axioms except efficiency.

(ii) The value defined by (5.5) satisfies all axioms except weak no advantageous reallocation.

(iii) The value defined by (5.6) satisfies all axioms except anonymity and weak desirability.

(iv) The ED value satisfies all axioms except proportional loss under separatorization.

(v) The value on $\mathcal{G}_{nz}^{\geq 3}$ defined for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - \left(v(\{i\}) - \frac{1}{n} \sum_{j\in N} v(\{j\}) \right) \left(\frac{1}{2} \right)^{\left|\sum\limits_{j\in N} v(\{j\})\right|}$$

satisfies all axioms except weak linearity.

(vi) The value $\psi(N,v) = 2EPSD(N,v) - PD(N,v)$ for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ satisfies all axioms of Theorem 5.4 except superadditive monotonicity.

Theorem 5.6:

(i) The value $\psi(N,v) = \alpha ED(N,v) + (1-\alpha)ESD(N,v)$ for all $(N,v) \in \mathcal{G}_{nz}^{N}$, satisfies all axioms except proportional loss under separatorization.

(ii) The value defined by (5.6) satisfies all axioms except the α-egalitarian inessential game property.

(iii) The value $\psi_i(N,v) = (1-\alpha)v(\{i\}) + \frac{\alpha}{n}\sum_{j\in N} v(\{j\})$ for all $(N,v) \in \mathcal{G}_{nz}^{N}$ and $i \in N$, satisfies all axioms except efficiency.

Theorem 5.7:

(i) The ED value satisfies all axioms except proportional loss under separatorization.

(ii) The value defined by (5.6) satisfies all axioms except α-reasonable lower bound.

(iii) The value $\psi_i(N,v) = \frac{1}{n}\sum_{j\in N} v(\{j\})$ for all $(N,v) \in \mathcal{G}_{nz}^{N}$ and $i \in N$, satisfies all axioms except efficiency.

5.7 Proofs

Let us denote $K(v) = \sum_{j\in N} v(\{j\})$ for any $(N,v) \in \mathcal{G}_{nz}^{N}$. If no ambiguity is possible, we use K instead of $K(v)$.

Proof of Lemma 5.1. Suppose that $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$. Denote $D(N,v) = \{i \in N \mid i \text{ is a separator in } (N,v)\}$. We proceed by induction on the decreasing cardinality of the set $D(N,v)$.

Initialization. For $|D(N,v)| = |N|$, i.e. all players are separators, (N,v) is an additive game. Then $\psi = \varphi$ by hypothesis. There is no game in which $|D(N,v)| = |N| - 1$, because if $|N| - 1$ players are separators then the nth one is also a separator. Therefore, $\psi = \varphi$ holds for $|D(N,v)| \geq |N| - 1$.

Induction hypothesis. Suppose that $\psi(N,v) = \varphi(N,v)$ for all games $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ such that $|D(N,v)| \geq d$, for $0 < d \leq |N| - 1$.

Induction step. Consider any game $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$ such that $|D(N,v)| = d - 1$. Since $d \leq |N| - 1$, and thus $|N \backslash D(N,v)| \geq 2$. Let h, l be two distinct players in $N \backslash D(N,v)$. For any $i, j \in N \backslash \{h\}$, by proportional loss under separatorization of ψ and φ,

$$\frac{\psi_i(N,v) - \psi_i(N,v^h)}{v(\{i\})} = \frac{\psi_j(N,v) - \psi_j(N,v^h)}{v(\{j\})}, \tag{5.7}$$

and

$$\frac{\varphi_i(N,v) - \varphi_i(N,v^h)}{v(\{i\})} = \frac{\varphi_j(N,v) - \varphi_j(N,v^h)}{v(\{j\})}. \tag{5.8}$$

Since h is a separator in (N,v^h) and not a separator in (N,v), and $D(N,v) \subset D(N,v^h)$, then $|D(N,v^h)| \geq |D(N,v)| + 1 = d$ [5]. The induction hypothesis then implies that

$$\psi_k(N,v^h) = \varphi_k(N,v^h), \text{ for all } k \in N. \tag{5.9}$$

Subtracting (5.8) from (5.7) and using (5.9) yields

$$\psi_i(N,v) - \varphi_i(N,v) = \frac{v(\{i\})}{v(\{j\})}[\psi_j(N,v) - \varphi_j(N,v)].$$

The above equality similarly holds for all $i,j \in N \backslash \{l\}$. Since $|N| \geq 3$, this equality holds for all $i,j \in N$. Then, summing this equality over $i \in N$ and using efficiency, we obtain

$$v(N) - v(N) = \frac{\sum_{i\in N} v(\{i\})}{v(\{j\})}[\psi_j(N,v) - \varphi_j(N,v)].$$

Since $\frac{\sum_{i\in N} v(\{i\})}{v(\{j\})} \neq 0$ for all $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$, it immediately follows that $\psi_j(N,v) = \varphi_j(N,v)$ for all $j \in N$. □

Proof of Lemma 5.2. Let ψ be a value on $\mathcal{G}_{nz}^{\geq 3}$ satisfying efficiency and proportional loss under separatorization. We first present two claims on ψ.

Claim 5.1. For any $h \in N$, $i \in N \backslash \{h\}$ and any nonempty $S \subseteq N \backslash \{i,h\}$,

$$\psi_i(N,v) - \psi_i(N,v^h) = \tfrac{v(\{i\})}{K - v(\{h\})}[v(N) - \psi_h(N,v) - v^h(N) + \psi_h(N,v^h)], \tag{5.10}$$

$$\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{h\}}) = \tfrac{v(\{i\})}{K - v(\{h\})}[-\psi_h(N,v^S) + \psi_h(N,v^{S\cup\{h\}})]. \tag{5.11}$$

Proof. Let $(N,v) \in \mathcal{G}_{nz}^N$, $h \in N$ and $i,j \in N \backslash \{h\}$. By proportional loss under separatorization, we have

$$\psi_j(N,v) - \psi_j(N,v^h) = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^h)].$$

Summing this equality over $j \in N \backslash \{h\}$ and using efficiency, we have

$$v(N) - \psi_h(N,v) - [v^h(N) - \psi_h(N,v^h)] = \tfrac{\sum_{j\in N\backslash\{h\}} v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^h)],$$

which implies (5.10).

[5]The inequality holds since making a player who was not a separator into a separator, may cause other players to lose their status.

Pick any nonempty $S \subseteq N$, and consider $(N, v^S) \in \mathcal{G}_{nz}^{\geq 3}$. Since $v^S(N) = \sum_{k \in N} v(\{k\}) = K$ and $v^S(\{k\}) = v(\{k\})$ for all $k \in N$, then (5.11) is implied by (5.10) applied to (N, v^S). $\square$

Claim 5.2. For all $S \subseteq N$ with $1 \leq |S| \leq n-1$, $\psi(N, v^S) = \psi(N, v^N)$.

Proof. The assertion is obtained by induction on the number of separators, again in decreasing order.

Initialization. Since $(N, v^{N \setminus \{h\}}) = (N, v^N)$ for any $h \in N$, we conclude that $\psi(N, v^S) = \psi(N, v^N)$ for all $S \subseteq N$ with $|S| = n-1$,

Induction hypothesis. Assume that $\psi(N, v^T) = \psi(N, v^N)$ holds for all $T \subseteq N$ with $|T| = t$, for some $2 \leq t \leq n-1$.

Induction step. Consider $(N, v^S) \in \mathcal{G}_{nz}^{\geq 3}$ and $S \subsetneq N$ such that $|S| = t-1$. Take $j \in N \setminus S$ and $i \in N \setminus (S \cup \{j\})$ (It is possible since $|S| \leq n-2$). We have

$$\begin{aligned}
& \psi_i(N, v^S) - \psi_i(N, v^{S \cup \{j\}}) \\
= \; & \frac{v(\{i\})}{\sum_{k \in N \setminus \{j\}} v(\{k\})} [-\psi_j(N, v^S) + \psi_j(N, v^{S \cup \{j\}})] \\
= \; & \frac{v(\{i\})}{\sum_{k \in N \setminus \{j\}} v(\{k\})} [-\psi_j(N, v^S) + \psi_j(N, v^N)] \\
= \; & \frac{v(\{i\})}{\sum_{k \in N \setminus \{j\}} v(\{k\})} [-\psi_j(N, v^S) + \psi_j(N, v^{S \cup \{i\}})] \\
= \; & \frac{v(\{i\})}{\sum_{k \in N \setminus \{j\}} v(\{k\})} [\frac{v(\{j\})}{\sum_{k \in N \setminus \{i\}} v(\{k\})} [\psi_i(N, v^S) - \psi_i(N, v^{S \cup \{i\}})]] \\
= \; & \frac{v(\{i\}) v(\{j\})}{\sum_{k \in N \setminus \{j\}} v(\{k\}) \sum_{k \in N \setminus \{i\}} v(\{k\})} [\psi_i(N, v^S) - \psi_i(N, v^{S \cup \{j\}})],
\end{aligned}$$

where the first and fourth equalities hold from (5.11), and the other three equalities hold by the induction hypothesis.

Since in general $\frac{v(\{i\}) v(\{j\})}{\sum_{k \in N \setminus \{j\}} v(\{k\}) \sum_{k \in N \setminus \{i\}} v(\{k\})} \neq 1$, we have $\psi_i(N, v^S) = \psi_i(N, v^{S \cup \{j\}})$ for all $i \in N \setminus (S \cup \{j\})$. For any $k \in S$, again by proportional loss under separatorization, we have $\frac{\psi_k(N, v^S) - \psi_k(N, v^{S \cup \{j\}})}{v(\{k\})} = \frac{\psi_i(N, v^S) - \psi_i(N, v^{S \cup \{j\}})}{v(\{i\})}$, which yields $\psi_k(N, v^S) = \psi_k(N, v^{S \cup \{j\}})$. Efficiency then implies that $\psi_j(N, v^S) = \psi_j(N, v^{S \cup \{j\}})$. Since there exists such a j for all $S \subsetneq N$, we conclude that $\psi(N, v^S) = \psi(N, v^{S \cup \{j\}}) \stackrel{\text{IH}}{=} \psi(N, v^N)$. $\square$

Based on Claims 5.1 and 5.2, we prove Lemma 5.2 as follows.

Proof of Lemma 5.2. For any $i \in N$ and $j \in N \setminus \{i\}$, Claim 5.2 together with (5.10) imply that

$$\begin{aligned}
& \psi_i(N, v) - \psi_i(N, v^N) \\
= \; & \frac{v(\{i\})}{\sum_{k \in N \setminus \{j\}} v(\{k\})} [v(N) - v^N(N) - \psi_j(N, v) + \psi_j(N, v^N)],
\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}&[K - v(\{j\})][\psi_i(N,v) - \psi_i(N,v^N)]\\ = \quad & v(\{i\})[v(N) - v^N(N) - (\psi_j(N,v) - \psi_j(N,v^N))].\end{aligned}$$

Summing the above equality over $j \in N\backslash\{i\}$ yields

$$\begin{aligned}&[(n-1)K - \sum_{j\in N\backslash\{i\}} v(\{j\})][\psi_i(N,v) - \psi_i(N,v^N)]\\ = \quad & v(\{i\})\Big[(n-1)[v(N) - v^N(N)] - \sum_{j\in N\backslash\{i\}} (\psi_j(N,v) - \psi_j(N,v^N))\Big].\end{aligned}$$

Using $\sum_{j\in N\backslash\{i\}}(\psi_j(N,v) - \psi_j(N,v^N)) = v(N) - \psi_i(N,v) - v^N(N) + \psi_i(N,v^N)$, which follows from efficiency, we have

$$\begin{aligned}&[(n-2)K + v(\{i\})][\psi_i(N,v) - \psi_i(N,v^N)]\\ = \quad & v(\{i\})\big[(n-2)[v(N) - v^N(N)] + [\psi_i(N,v) - \psi_i(N,v^N)]\big].\end{aligned}$$

Since $n-2 \neq 0$, it follows that

$$K[\psi_i(N,v) - \psi_i(N,v^N)] = v(\{i\})[v(N) - v^N(N)],$$

as desired. □

Proof of Theorem 5.1. It can easily be checked that any value of the form given in (5.4) satisfies the four axioms on $\mathcal{QA}_{nz}^{\geq 3}$. To prove the 'only if' part, let ψ be a value on $\mathcal{QA}_{nz}^{\geq 3}$ satisfying the four axioms. Let $(N,v) \in \mathcal{QA}_{nz}^{\geq 3}$ and let $i,j \in N$ be two fixed players. Without loss of generality, we assume that (N,v) is individually positive. Let $\varepsilon \in \mathbb{R}_+$ be any number such that $0 < \varepsilon < \min_{i\in N}\{v(\{i\})\}$. (The proof can be similarly written for the class of individually negative games, and then g should be a function $\mathbb{R}_- \times \mathbb{R} \to \mathbb{R}$.)

First, we consider the following quasi-additive games $v_t (t = 1,2,...,7)$ such that the worth of the grand coalition $v(N)$ and the sum of all stand-alone worths K are identical to those of (N,v).

(i) Consider the game $(N,v_1) \in \mathcal{QA}_{nz}^{\geq 3}$ defined by $v_1(\{i\}) = v(\{i\}) + v(\{j\}) - \varepsilon$, $v_1(\{j\}) = \varepsilon$, $v_1(\{k\}) = v(\{k\})$ for all $k \in N\backslash\{i,j\}$ and $v_1(N) = v(N)$. This involves a transfer from j to i. By weak no advantageous reallocation, we have

$$\psi_i(N,v) + \psi_j(N,v) = \psi_i(N,v_1) + \psi_j(N,v_1). \tag{5.12}$$

(ii) Consider the game $(N,v_2) \in \mathcal{QA}_{nz}^{\geq 3}$ defined by $v_2(\{i\}) = v(\{i\})$, $v_2(\{j\}) = K - v(\{i\}) - (n-2)\varepsilon$, $v_2(\{k\}) = \varepsilon$ for all $k \in N\backslash\{i,j\}$, and $v_2(N) = v(N)$.

This involves a transfer from the players in $N\backslash\{i,j\}$ to player j. By weak no advantageous reallocation applied to (N,v) and (N,v_2), we obtain

$$\sum_{k\in N\backslash\{i\}} \psi_k(N,v) = \sum_{k\in N\backslash\{i\}} \psi_k(N,v_2).$$

Efficiency then implies

$$\psi_i(N,v) = \psi_i(N,v_2). \tag{5.13}$$

(iii) Consider the game $(N,v_3) \in \mathcal{QA}_{nz}^{\geq 3}$ defined by $v_3(\{i\}) = v(\{j\})$, $v_3(\{j\}) = K - v(\{j\}) - (n-2)\varepsilon$, $v_3(\{k\}) = \varepsilon$ for all $k \in N\backslash\{i,j\}$, and $v_3(N) = v(N)$. This game is obtained by first switching roles between i and j in game (N,v) and then making a transfer similar to the one in case (ii). Let π be a permutation such that $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for all $k \in N\backslash\{i,j\}$.

Define the game $(N,v_4) \in \mathcal{QA}_{nz}^{\geq 3}$ by $v_4(\{i\}) = K - v(\{j\}) - (n-2)\varepsilon$, $v_4(\{j\}) = v(\{j\})$, $v_4(\{k\}) = \varepsilon$ for all $k \in N\backslash\{i,j\}$, and $v_4(N) = v(N)$. By weak no advantageous reallocation applied to (N,v) and (N,v_4), we obtain $\sum_{k\in N\backslash\{j\}} \psi_k(N,v) = \sum_{k\in N\backslash\{j\}} \psi_k(N,v_4)$. Efficiency then implies $\psi_j(N,v) = \psi_j(N,v_4)$. Notice that $(N,v_4) = (N,\pi v_3)$. By anonymity, $\psi_j(N,v_4) = \psi_{\pi(i)}(N,\pi v_3) = \psi_i(N,v_3)$. Therefore,

$$\psi_j(N,v) = \psi_i(N,v_3). \tag{5.14}$$

(iv) Consider the game $(N,v_5) \in \mathcal{QA}_{nz}^{\geq 3}$ defined by $v_5(\{i\}) = v(\{i\}) + v(\{j\}) - \varepsilon$, $v_5(\{j\}) = K - v(\{i\}) - v(\{j\}) - (n-3)\varepsilon$, $v_5(\{k\}) = \varepsilon$ for all $k \in N\backslash\{i,j\}$, and $v'_{ij}(N) = v(N)$. This involves a transfer from the players in $N\backslash\{i,j\}$ to player j given game (N,v_1). By weak no advantageous reallocation applied to (N,v_1) and (N,v_5), we obtain $\sum_{k\in N\backslash\{i\}} \psi_k(N,v_1) = \sum_{k\in N\backslash\{i\}} \psi_k(N,v_5)$. Efficiency then implies

$$\psi_i(N,v_1) = \psi_i(N,v_5). \tag{5.15}$$

(v) Consider the game $(N,v_6) \in \mathcal{QA}_{nz}^{N}$ defined by $v_6(\{j\}) = K - (n-1)\varepsilon$, $v_6(\{k\}) = \varepsilon$ for all $k \in N\backslash\{j\}$, and $v_6(N) = v(N)$. This involves a transfer from i to j given game (N,v_5). Let $(N,v_7) \in \mathcal{QA}_{nz}^{\geq 3}$ be the game defined by $v_7(\{i\}) = K - (n-1)\varepsilon$, $v_7(\{k\}) = \varepsilon$ for all $k \in N\backslash\{i\}$, and $v_7(N) = v(N)$. Clearly, $(N,v_7) = (N,\pi v_6)$ for the permutation such that $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for all $k \in N\backslash\{i,j\}$. By anonymity, we obtain $\psi_i(N,v_6) = \psi_j(N,v_7)$. On the other hand, applying weak no advantageous reallocation to (N,v_7) and (N,v_1), and then using efficiency, we obtain $\psi_j(N,v_7) = \psi_j(N,v_1)$. Therefore,

$$\psi_j(N,v_1) = \psi_i(N,v_6). \tag{5.16}$$

Next, based on (5.12)-(5.16), we derive the formula of $\psi_i(N,v)$. Substituting (5.13)-(5.16) into (5.12), we have

$$\psi_i(N,v_2)+\psi_i(N,v_3)=\psi_i(N,v_5)+\psi_i(N,v_6). \tag{5.17}$$

Each game used in (5.17) is uniquely determined by four parameters: the worth of $\{i\}$, the number ε (which determines the stand-alone worths of players $k\in N\backslash\{i,j\}$), the sum of stand-alone worths K (which, with ε, determines the stand-alone worth of j), and the worth of the grand coalition $v(N)$. For such game (N,v_0), given $\varepsilon,K,v(N)$, let $F(v_0(\{i\}))=\psi_i(N,v_0)$. Clearly,

$$\begin{cases}\psi_i(N,v_2)=F(v(\{i\})),\\ \psi_i(N,v_3)=F(v(\{j\})),\\ \psi_i(N,v_5)=F(v(\{i\})+v(\{j\})-\varepsilon),\\ \psi_i(N,v_6)=F(\varepsilon).\end{cases} \tag{5.18}$$

Here, only $v(\{i\})$ and $v(\{j\})$ are variables, and $(v(\{k\}))_{k\in N\backslash\{i,j\}},\varepsilon,K,v(N)$ are constants.

Since we can take any worth $v(\{i\})>0$ for given $\varepsilon,K,v(N)$, we consider $F(v_0(\{i\}))$ to be a function on $\mathbb{R}_+$. For $x-\varepsilon>0$, we define a funcion $f:\{x\in\mathbb{R}_+|x>\varepsilon\}\to\mathbb{R}$ by

$$f(c-\varepsilon)=F(c)-F(\varepsilon). \tag{5.19}$$

Note that (5.17) can be rewritten as

$$\psi_i(N,v_2)-\psi_i(N,v_6)+\psi_i(N,v_3)-\psi_i(N,v_6)=\psi_i(N,v_5)-\psi_i(N,v_6).$$

Taking (5.18) and (5.19) into account, we can then write

$$\begin{aligned}&\left[F(v(\{i\}))-F(\varepsilon)\right]+\left[F(v(\{j\}))-F(\varepsilon)\right]\\ =\;&F(v(\{i\})+v(\{j\})-\varepsilon)-F(\varepsilon),\end{aligned}$$

which is equivalent to

$$f(v(\{i\})-\varepsilon)+f(v(\{j\})-\varepsilon)=f(v(\{i\})+v(\{j\})-2\varepsilon)$$

Here note that only $v(\{i\})$ and $v(\{j\})$ are variables. Since $v(\{i\})+v(\{j\})-2\varepsilon=[v(\{i\})-\varepsilon]+[v(\{j\})-\varepsilon]$, f is additive. By continuity of ψ, f is continuous. Therefore, applying Theorem 2 on the conditional Cauchy equation of Aczél and Erdős (1965) to f, and then using Corollary 3.1.9, p.51, of Eichhorn (1978) on Cauchy's equation, yield that there exists a constant f_0 such that

$$f(c-\varepsilon)=(c-\varepsilon)f_0. \tag{5.20}$$

Substituting (5.20) into (5.19) and taking $c = v(\{i\})$, it follows that

$$(v(\{i\}) - \varepsilon)f_0 = F(v(\{i\})) - F(\varepsilon).$$

We obtain

$$\begin{aligned}\psi_i(N, v) &= \psi_i(N, v_2)\\ &= F(v(\{i\}))\\ &= f(v(\{i\}) - \varepsilon) + F(\varepsilon)\\ &= (v(\{i\}) - \varepsilon)f_0 + F(\varepsilon).\end{aligned} \tag{5.21}$$

where the first equality follows from (5.13), the second from (5.18), the third from (5.19), and the last from (5.20).

Note that (5.21) holds for all $i \in N$. Summing up these equations over all $i \in N$ and using efficiency, we obtain

$$v(N) = (K - n\varepsilon)f_0 + nF(\varepsilon).$$

It follows that

$$F(\varepsilon) = \frac{v(N)}{n} - \frac{K}{n}f_0 + \varepsilon f_0.$$

The above equation and (5.21) yield

$$\psi_i(N, v) = v(\{i\})f_0 + \frac{v(N)}{n} - \frac{K}{n}f_0. \tag{5.22}$$

This equation was obtained for any fixed ε. Then f_0 might depend on ε. We show that f_0 is independent of ε.

Take any two positive numbers $\varepsilon_1, \varepsilon_2 < \min_{i\in N}\{v(\{i\})\}$. Suppose f_0 depends on ε and denote $f_0(\varepsilon_1)$ and $f_0(\varepsilon_2)$, respectively. (5.22) yields $v(\{i\})f_0(\varepsilon_1) + \frac{v(N)}{n} - \frac{K}{n}f_0(\varepsilon_1) = \psi_i(N, v) = v(\{i\})f_0(\varepsilon_2) + \frac{v(N)}{n} - \frac{K}{n}f_0(\varepsilon_2)$, and thus it must be that $(v(\{i\}) - \frac{K}{n})(f_0(\varepsilon_1) - f_0(\varepsilon_2)) = 0$. It is possible to take $v(\{i\})$ with $v(\{i\}) - \frac{K}{n} \neq 0$, and thus $f_0(\varepsilon_1) = f_0(\varepsilon_2)$. Hence, if $v(\{i\}) - \frac{K}{n} \neq 0$ then $f_0(\varepsilon_1) = f_0(\varepsilon_2)$; otherwise, continuity also implies $f_0(\varepsilon_1) = f_0(\varepsilon_2)$. This means that the number f_0 does not depend on ε.

This argument depends on the choices of K and $v(N)$. Hence we denote $f_0 = f_0(K, v(N))$. Then we consider a function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by $f_0(K, v(N)) = \frac{v(N)}{K} - \frac{1}{K}g(K, v(N))$. Using this function, (5.22) can be rewritten as

$$\begin{aligned}\psi_i(N, v) &= \frac{v(\{i\})v(N)}{K} - \frac{v(\{i\})}{K}g(K, v(N)) + \frac{v(N)}{|N|} - \frac{Kv(N)}{nK} + \frac{Kg(K, v(N))}{nK}\\ &= \frac{v(\{i\})}{K}v(N) - \left(\frac{v(\{i\})}{K} - \frac{1}{n}\right)g(K, v(N)),\end{aligned}$$

as desired. □

Proof of Theorem 5.2. Since it is obvious that $\psi^\alpha = \alpha EPSD + (1-\alpha)PD$, $\alpha \in \mathbb{R}$, satisfies efficiency, anonymity, and weak linearity, we only show that ψ satisfies proportional loss under separatorization and weak no advantageous reallocation. For any $(N,v) \in \mathcal{G}_{nz}^{\geq 3}$, $h \in N$ and $i \in N\backslash\{h\}$, using (5.1) and the definition of (N,v^h), we have

$$\begin{aligned}\varphi_i^\alpha(N,v^h) &= \frac{\alpha}{n}\sum_{j\in N} v^h(\{j\}) + \frac{v^h(\{i\})}{\sum_{j\in N} v^h(\{j\})}[v^h(N) - \sum_{j\in N}\alpha v^h(\{j\})] \\ &= \frac{\alpha}{n}K(v) + \frac{v(\{i\})}{K(v)}[K(v) - \alpha K(v)].\end{aligned}$$

Subtracting the above equation from (5.1) for the game (N,v), we have

$$\varphi_i^\alpha(N,v) - \varphi_i^\alpha(N,v^h) = \frac{v(\{i\})}{K(v)}[v(N) - K(v)].$$

It follows that

$$\frac{\varphi_i^\alpha(N,v) - \varphi_i^\alpha(N,v^h)}{v(\{i\})} = \frac{1}{K(v)}[v(N) - K(v)],$$

which shows that proportional loss under separatorization is satisfied.

To show that φ^α satisfies weak no advantageous reallocation, let $(N,v),(N,w) \in \mathcal{QA}_{nz}^{\geq 3}$ and $T \subseteq N$ be such that $v(N) = w(N)$, $\sum_{i\in T} v(\{i\}) = \sum_{i\in T} w(\{i\})$ and $v(\{i\}) = w(\{i\})$ for all $i \in N\backslash T$. Clearly, $K(v) = K(w)$. Then, using (5.1),

$$\begin{aligned}\sum_{i\in T}\varphi_i^\alpha(N,v) &= \sum_{i\in T}[\frac{\alpha}{n}K(v) + \frac{v(\{i\})}{K(v)}[v(N) - \alpha K(v)]] \\ &= \frac{\alpha t}{n}K(v) + \frac{\sum_{i\in T} v(\{i\})}{K(v)}[v(N) - \alpha K(v)] \\ &= \frac{\alpha t}{n}K(w) + \frac{\sum_{i\in T} w(\{i\})}{K(w)}[w(N) - \alpha K(w)] \\ &= \sum_{i\in T}\varphi_i^\alpha(N,w),\end{aligned}$$

which shows that weak no advantageous reallocation is satisfied.

It remains to prove the 'only if' part. Let ψ be a value on $\mathcal{G}_{nz}^{\geq 3}$ that satisfies the five axioms.

First, consider any game $(N,v) \in \mathcal{QA}_{nz}^{\geq 3}$ and $(N,v^N) \in \mathcal{A}_{nz}^{\geq 3}$. From Lemma 5.2,

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{v(\{i\})}{K}v(N) - v(\{i\}) \text{ for all } i \in N. \tag{5.23}$$

Since (N,v^N) is an additive game, $\psi_i(N,v^N)$ can be seen as a function of the stand-alone worths $v(\{i\})$, $i \in N$. Moreover, since the right-hand side of (5.23) only has the terms of $v(S)$ with $|S| = 1, n$, we obtain from (5.23) that $\psi_i(N,v)$ has the term $\frac{v(\{i\})}{K}v(N)$, but no terms of $v(S)$, where $S \subseteq N$ with $1 < |S| < n$. This implies that $\psi_i(N,v)$ does not depend on $v(S)$, $S \subseteq N$, $1 < |S| < n$, and is continuous with respect to $v(N)$. Hence, from Remark 5.3 and Theorem 5.1, $\psi_i(N,v)$ and $\psi_i(N,v^N)$ have the form of (5.4). Substituting them into (5.23), we obtain for every $i \in N$,

$$\begin{aligned}
& \psi_i(N,v) - \psi_i(N,v^N) - \frac{v(\{i\})}{K}v(N) + v(\{i\}) \\
= \; & \frac{v(\{i\})v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n}\right) g(K,v(N)) - \frac{v(\{i\})v^N(N)}{K} \\
& + \left(\frac{v(\{i\})}{K} - \frac{1}{n}\right) g(K,v^N(N)) - \frac{v(\{i\})v(N)}{K} + v(\{i\}) \\
= \; & -\left(\frac{v(\{i\})}{K} - \frac{1}{n}\right) (g(K,v(N)) - g(K,v^N(N))) \\
= \; & 0,
\end{aligned} \tag{5.24}$$

where in the second equality we use $v^N(N) = K$, and the last equality follows from (5.23).

To obtain the formula of $\psi_i(N,v)$, $(N,v) \in \mathcal{QA}_{nz}^{\geq 3}$, we consider two cases:

(i) Suppose that $(N,v) \in \mathcal{QA}_{nz}^{N}$ is such that $v(\{i\}) \neq v(\{j\})$ for some $i,j \in N$. It must be that $\frac{v(\{h\})}{K} \neq \frac{1}{n}$ for some $h \in N$. Then, from (5.24) we obtain $g(K,v(N)) = g(K,v^N(N))$. This means that $g : \mathbb{R}\backslash\{0\} \times \mathbb{R} \to \mathbb{R}$ is a constant function with respect to its second argument for each K since $v^N(N) = K$. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be such that $f(x) = g(x,y)$ for all $x \in \mathbb{R}\backslash\{0\}$ and $y \in \mathbb{R}$. Then (5.4) can be written as

$$\psi_i(N,v) = \frac{v(\{i\})v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n}\right) f(K). \tag{5.25}$$

Consider any $(N,v), (N,w) \in \mathcal{QA}_{nz}^{\geq 3}$ and $a \in \mathbb{R}$ such that $(N, av+w) \in \mathcal{QA}_{nz}^{\geq 3}$ and there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$. By weak linearity, $\psi_i(N,av+w) = a\psi_i(N,v) + \psi_i(N,w)$ for all $i \in N$. Using (5.25), this yields that $f(K(av+w)) = af(K(v)) + f(K(w))$, which implies that f satisfies linearity on $\mathbb{R}\backslash\{0\}$. Hence, $f(K) = \alpha K$, where α is an arbitrary constant. Therefore, using (5.25), we have

$$\begin{aligned}
\psi_i(N,v) &= \frac{v(\{i\})v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n}\right) \alpha K \\
&= \frac{v(\{i\})v(N)}{K} - \alpha v(\{i\}) + \frac{1}{n}\alpha K \\
&= \frac{1}{n}\sum_{j\in N} \alpha v(\{j\}) + \frac{v(\{i\})}{K}[v(N) - \sum_{j\in N}\alpha v(\{j\})],
\end{aligned}$$

which equals to Formule (5.1) of $\varphi^\alpha(N,v)$.

(ii) Suppose that $(N,v) \in \mathcal{QA}_{nz}^{\geq 3}$ is such that $v(\{i\}) = v(\{j\})$ for all $i,j \in N$. Then, by (5.4) we have $\psi_i(N,v) = \frac{v(N)}{n}$, which also satisfies (5.1).

Second, consider any game $(N, v) \in \mathcal{G}_{nz}^{\geq 3}$. Since (N, v^N) is an additive game, by (5.1) applied to (N, v^N), we have $\psi_i(N, v^N) = \frac{\alpha}{n}K + \frac{v(\{i\})}{K}(K - \alpha K) = \frac{\alpha}{n}\sum_{j \in N} v(\{j\}) + (1-\alpha)v(\{i\})$. Substituting this equation into (5.3) from Lemma 5.2, we obtain $\psi_i(N, v) = \psi_i(N, v^N) + \frac{v(\{i\})}{K}v(N) - v(\{i\}) = \frac{\alpha}{n}K + (1-\alpha)v(\{i\}) + \frac{v(\{i\})}{K}v(N) - v(\{i\})$, which coincides with (5.2), and thus $\psi(N, v) = \alpha EPSD(N, v) + (1-\alpha)PD(N, v)$. Then by Lemma 5.1, this value is uniquely determined. □

Proof of Theorem 5.3. For the 'if' part, we already know that $\varphi^\alpha = \alpha EPSD + (1-\alpha)PD$ satisfies efficiency, anonymity, weak no advantageous reallocation, proportional loss under separatorization, and weak linearity. We show that φ^α also satisfies superadditive monotonicity if $\alpha \in [0, \frac{n}{n-1}]$. Let $(N, v) \in \mathcal{G}_{nz}^{\geq 3}$ be an arbitrary superadditive and monotone game. Since $v(N) \geq \sum_{j \in N} v(\{j\})$, by (5.2) we have $\psi_i(N, v) \geq \frac{\alpha}{n}\sum_{j \in N} v(\{j\}) + (1-\alpha)v(\{i\}) > \frac{\alpha}{n}v(\{i\}) + (1-\alpha)v(\{i\}) = \left(1 - \frac{n-1}{n}\alpha\right) v(\{i\}) \geq 0$. Hence, ψ satisfies superadditive monotonicity.

It remains to prove the 'only if' part. Let ψ be a value on $\mathcal{G}_{nz}^{\geq 3}$ satisfying the six axioms. From Theorem 5.2, there exists $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1-\alpha)PD$. We must show that α belongs to $[0, \frac{n}{n-1}]$. Suppose, by contradiction, that $\alpha \notin [0, \frac{n}{n-1}]$. We distinguish the following two cases.

(i) Suppose that $\alpha < 0$. Consider an additive game $(N, v) \in \mathcal{A}_{nz}^{\geq 3}$, where $v(\{i\}) = 1$ and $v(\{j\}) = 1 - \frac{\alpha}{n-1} - \frac{n}{(n-1)\alpha}$ for all $j \in N \backslash \{i\}$. Clearly, this game is superadditive and monotone since $\frac{\alpha}{n-1} + \frac{n}{(n-1)\alpha} = \frac{\alpha^2+n}{(n-1)\alpha} < 0$. By Theorem 5.2 and (N, v) being additive, $\psi_i(N, v) = \frac{\alpha}{n}\sum_{j \in N} v(\{j\}) + (1-\alpha)v(\{i\}) = \frac{\alpha}{n}(1 + (n-1) - \alpha - \frac{n}{\alpha}) + 1 - \alpha = \frac{\alpha+(n-1)\alpha-\alpha^2}{n} - 1 + 1 - \alpha = -\frac{\alpha^2}{n} < 0$, which contradicts superadditive monotonicity.

(ii) Suppose that $\alpha > \frac{n}{n-1}$. Consider an additive game $(N, v) \in \mathcal{A}_{nz}^{\geq 3}$ such that $v(\{i\}) = 1 + \frac{2n}{(n-1)\alpha - n}$ and $v(\{j\}) = 1$ for all $j \in N \backslash \{i\}$. Also this game is superadditive and monotone. In this case, $\psi_i(N, v) = \frac{\alpha}{n}(n + \frac{2n}{(n-1)\alpha-n}) + (1-\alpha)(1 + \frac{2n}{(n-1)\alpha-n}) = \alpha + \frac{2\alpha}{(n-1)\alpha-n} + 1 - \alpha + \frac{2n(1-\alpha)}{(n-1)\alpha-n} = 1 + \frac{2\alpha+2n(1-\alpha)}{(n-1)\alpha-n} = -1 < 0$, which contradicts superadditive monotonicity.

□

Proof of Theorem 5.4. It is easy to check that $\psi = \alpha EPSD + (1-\alpha)PD$, $\alpha \in [0, 1]$, satisfies the six axioms. For the uniqueness, Theorem 5.3 and Lemma 5.3 imply that we have to show $\alpha \leq 1$, which follows immediately from (5.2) and weak desirability. □

Proof of Lemma 5.3. Let ψ be a value on $\mathcal{G}_{nz}^{\geq 3}$ satisfying efficiency, proportional loss under separatorization, weak no advantageous reallocation, and weak desirability. Let π be a permutation on N. First, consider any two games $(N, v), (N, w) \in \mathcal{A}_{nz}^{\geq 3}$ such that $(N, w) = (N, \pi v)$. Without loss of generality, we assume that (N, v) is individually positive. We distinguish the following three cases with respect to the players:

(i) $\pi(i) = i$. In this case, $v(\{i\}) = w(\{i\})$ and $\sum_{k\in N\setminus\{i\}} v(\{k\}) = \sum_{k\in N\setminus\{i\}} w(\{k\})$. By weak no advantageous reallocation, $\sum_{k\in N\setminus\{i\}} \psi_k(N,v) = \sum_{k\in N\setminus\{i\}} \psi_k(N,w)$. Efficiency then implies that $\psi_i(N,v) = \psi_i(N,w) = \psi_{\pi(i)}(N,\pi v)$.

(ii) $\pi(i) \neq i$ and $v(\{i\}) < \frac{K}{2}$. We consider a game $(N,v') \in \mathcal{A}_{nz}^{\geq 3}$ such that $v'(\{i\}) = v'(\{j\}) = v(\{i\})$ and $\sum_{k\in N\setminus\{i,j\}} v'(\{k\}) = K - 2v(\{i\})$, where $j = \pi(i)$. By weak no advantageous reallocation applied to $(N,v), (N,v')$ and $N\setminus\{i\}$, we have $\sum_{k\in N\setminus\{i\}} \psi_k(N,v) = \sum_{k\in N\setminus\{i\}} \psi_k(N,v')$. This together with efficiency imply that $\psi_i(N,v) = \psi_i(N,v')$. On the other hand, by weak no advantageous reallocation applied to $(N,w), (N,v')$ and $N\setminus\{j\}$, we have $\sum_{k\in N\setminus\{j\}} \psi_k(N,w) = \sum_{k\in N\setminus\{j\}} \psi_k(N,v')$. Efficiency then implies that $\psi_j(N,w) = \psi_j(N,v')$. Moreover, since $v'(S\cup\{i\}) = v'(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$, weak desirability implies that $\psi_i(N,v') = \psi_j(N,v')$. Therefore, $\psi_i(N,v) = \psi_j(N,w) = \psi_{\pi(i)}(N,\pi v)$.

(iii) $\pi(i) \neq i$ and $v(\{i\}) \geq \frac{K}{2}$. Since $|N| \geq 3$, there exists at most one such player. Applying cases (i) and (ii) to all other players $j \in N\setminus\{i\}$, we have that $\psi_j(N,v) = \psi_{\pi(j)}(N,\pi v)$ for all $j \in N\setminus\{i\}$. Efficiency then implies that $\psi_i(N,v) = \psi_{\pi(i)}(N,\pi v)$.

The above three cases show that if a value ψ on $\mathcal{A}_{nz}^{\geq 3}$ satisfies efficiency, weak no advantageous reallocation, and weak desirability, then it also satisfies anonymity. From Lemma 5.2, efficiency and proportional loss under separatorization together imply (5.3), and thus $\psi_i(N,v) = \psi_i(N,v^N) + \frac{v(\{i\})}{K} v(N) - v(\{i\}) = \psi_{\pi(i)}(N,\pi v^N) + \frac{\pi v(\{\pi(i)\})}{K} \pi v(N) - \pi v(\{\pi(i)\}) = \psi_{\pi(i)}(N,\pi v)$ since $\pi v^N = (\pi v)^N$. Thus, ψ satisfies anonymity on $\mathcal{G}_{nz}^{\geq 3}$. □

Proof of Lemma 5.4. It is easy to check that the assertion holds for $|N| = 2$. For $|N| \geq 3$, the proof is similar to the proof of Lemma 5.1 except the induction step, which now is as follows.

Induction step. Consider any game $(N,v) \in \mathcal{G}_{nz}^N$ such that $|D(N,v)| = d - 1$. Since $d < n - 1$, then $|N\setminus D(N,v)| \geq 2$.

First, consider any $i \in N\setminus D(N,v)$ and any $j \in D(N,v)$. Obviously, $|D(N,v^i)| \geq |D(N,v)| + 1 = d$ and $(N,v) = (N,v^j)$. Proportional balanced contributions under separatorization and the induction hypothesis imply that

$$\begin{aligned}
\psi_j(N,v) &= \psi_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^j)] \\
&= \psi_j(N,v^i) \\
&= \varphi_j(N,v^i) \\
&= \varphi_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}[\varphi_i(N,v) - \varphi_i(N,v^j)]
\end{aligned}$$

$$= \varphi_j(N,v), \tag{5.26}$$

where the first and the last equalities follow from proportional balanced contributions under separatorization, and the third equality holds by the induction hypothesis.

Next consider two distinct players $i, k \in N \backslash D(N,v)$. Again, proportional balanced contributions under separatorization and the induction hypothesis imply that

$$\begin{aligned}
\psi_k(N,v) &= \psi_k(N,v^i) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^k)] \\
&= \varphi_k(N,v^i) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v^k)] \\
&= \varphi_k(N,v) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)],
\end{aligned}$$

where again the first and the last equalities follow from proportional balanced contributions under separatorization, and the second equality holds by the induction hypothesis.

Thus

$$\psi_k(N,v) - \varphi_k(N,v) = \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)].$$

Summing the above equality over $k \in N \backslash D(N,v)$ and then using efficiency, we obtain that

$$\sum_{k \in N \backslash D(N,v)} (\psi_k(N,v) - \varphi_k(N,v)) = \sum_{k \in N \backslash D(N,v)} \left(\frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)] \right)$$

$$\Leftrightarrow$$

$$\begin{aligned}
& v(N) - \sum_{j \in D(N,v)} \psi_j(N,v) - v(N) + \sum_{j \in D(N,v)} \varphi_j(N,v) \\
&= \sum_{k \in N \backslash D(N,v)} \left(\frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)] \right)
\end{aligned}$$

$$\Leftrightarrow$$

$$0 = \frac{\psi_i(N,v) - \varphi_i(N,v)}{v(\{i\})} \sum_{k \in N \backslash D(N,v)} v(\{k\}),$$

where the second equivalence follows from (5.26). Thus, since $v(\{k\}) \neq 0$ for all $k \in N$, $\psi_i(N,v) = \varphi_i(N,v)$ for any $i \in N \backslash D(N,v)$. □

Proof of Lemma 5.5. Let ψ be a value on $\mathcal{G}^N_{nz}$ satisfying the two axioms. For any $(N,v) \in \mathcal{G}^N_{nz}$ and any $i, j \in N$, by proportional balanced contributions under separatorization, we have $\psi_j(N,v) - \psi_j(N,v^i) = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^j)]$. Summing this equality over $j \in N\backslash\{i\}$ and using efficiency, we have

$$\begin{aligned}&v(N) - \psi_i(N,v) - [v^i(N) - \psi_i(N,v^i)]\\&= \frac{\sum_{j\in N\backslash\{i\}} v(\{j\})}{v(\{i\})}\psi_i(N,v) - \frac{1}{v(\{i\})}\sum_{j\in N\backslash\{i\}} v(\{j\})\psi_i(N,v^j).\end{aligned}$$

It follows that

$$\psi_i(N,v)\left(\frac{K}{v(\{i\})}\right) = v(N) - v^i(N) + \psi_i(N,v^i) + \frac{1}{v(\{i\})}\sum_{j\in N\backslash\{i\}} v(\{j\})\psi_i(N,v^j),$$

and thus

$$\psi_i(N,v) = \frac{v(\{i\})}{K}[v(N) - v^i(N)] + \sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^j). \tag{5.27}$$

Next, we show that $\psi(N,v^S) = \psi(N,v^N)$ for all $S \subseteq N$ with $1 \leq |S| \leq n-1$. We use an induction on the number of separators.

Initialization. Since $(N,v^{N\backslash\{h\}}) = (N,v^N)$ for all $h \in N$, then $\psi(N,v^S) = \psi(N,v^N)$ for all $S \subseteq N$ with $|S| = n-1$.

Induction hypothesis. Assume that $\psi(N,v^T) = \psi(N,v^N)$ holds for all $T \subseteq N$ with $|T| = t$ for some $2 \leq t \leq n-1$.

Induction step. Consider $(N,v^S) \in \mathcal{G}^N$ and $S \subsetneq N$ such that $|S| = t-1$. Let $i,k \in N\backslash S$ be two distinct players. We have

$$\begin{aligned}&\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{k\}})\\&= \sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^{S\cup\{j\}}) - \sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^{S\cup\{k,j\}})\\&= \sum_{j\in N}\frac{v(\{j\})}{K}[\psi_i(N,v^{S\cup\{j\}}) - \psi_i(N,v^{S\cup\{k,j\}})]\\&= \sum_{j\in N}\frac{v(\{j\})}{K}[\psi_i(N,v^N) - \psi_i(N,v^N)]\\&= 0,\end{aligned}$$

where the first equality holds from (5.27), and the third equality holds by the induction hypothesis.

Hence,

$$\psi_i(N,v^S) = \psi_i(N,v^{S\cup\{k\}}) = \psi_i(N,v^N) \text{ for all } i \in N\backslash S, \tag{5.28}$$

where the second equality holds by the induction hypothesis.

To prove this equality also for all $j \in S$, pick $i \in N\backslash S$ and $j \in S$. Proportional balanced contributions under separatorization implies that $\psi_j(N,v^S) - \psi_j(N,v^{S\cup\{i\}}) = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{j\}})] = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v^S) - \psi_i(N,v^S)] = 0$, where the second equality holds from (5.27).

Hence,

$$\psi_j(N, v^S) = \psi_j(N, v^{S\cup\{i\}}) = \psi_j(N, v^N) \quad \text{for all } j \in S,$$

where the second equality holds by the induction hypothesis.

Therefore, $\psi(N, v^S) = \psi(N, v^N)$ holds for all $S \subseteq N$ with $1 \leq |S| \leq n-1$. This, together with (5.27), yields the desired formula. □

Proof of Theorem 5.6. It is clear that φ^α satisfies the three axioms. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying the three axioms. For $|N| = 1$, $\psi = \varphi^\alpha$ holds from efficiency. Next, suppose that $|N| \geq 3$. By Lemma 5.2, efficiency and proportional loss under separatorization imply that ψ satisfies (5.3). Moreover, the α-egalitarian inessential game property implies that $\psi_i(N, v^N) = (1-\alpha)v(\{i\}) + \alpha\frac{v^N(N)}{n} = (1-\alpha)v(\{i\}) + \frac{\alpha}{n}\sum_{j\in N} v(\{j\})$. These two equations imply $\psi = \varphi^\alpha$. □

Proof of Theorem 5.7. It is clear that φ^α satisfies the three axioms. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying the three axioms. For any $(N, v) \in \mathcal{G}^N_{nz}$, consider a game $(N, w) \in \mathcal{G}^N_{nz}$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$, and $w(N) = \alpha\sum_{j\in N} w(\{j\}) = \alpha\sum_{j\in N} v(\{j\})$. From Lemma 5.2, efficiency and proportional loss under separatorization imply that $\psi_i(N, w) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} w(N) - w(\{i\}) + \psi_i(N, w^N) = (\alpha - 1)w(\{i\}) + \psi_i(N, w^N)$ for all $i \in N$. Since (N, w) is an α-essential game, α-reasonable lower bound gives that $\psi_i(N, w) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\})$. Hence,

$$(\alpha - 1)w(\{i\}) + \psi_i(N, w^N) = \psi_i(N, w) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\}) \quad \text{for all } i \in N,$$

and thus $\psi_i(N, w^N) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\}) + (1-\alpha)w(\{i\})$ for all $i \in N$. By efficiency applied to (N, w^N) implies that $w^N(N) = \sum_{i\in N}\psi_i(N, w^N) \geq \alpha\sum_{j\in N} w(\{j\}) + (1-\alpha)\sum_{j\in N} w(\{j\}) = \sum_{j\in N} w(\{j\}) = w^N(N)$, and thus these inequalities are equalities. Thus,

$$\psi_i(N, w^N) = \frac{\alpha}{n}\sum_{j\in N} w(\{j\}) + (1-\alpha)w(\{i\}).$$

Since $(N, v^N) = (N, w^N)$, then $\psi_i(N, v^N) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1-\alpha)v(\{i\})$. Again, by Lemma 5.2, we have

$$\begin{aligned}
\psi_i(N, v) &= \frac{v(\{i\})}{K}v(N) - v(\{i\}) + \psi_i(N, v^N) \\
&= \frac{v(\{i\})}{K}v(N) - v(\{i\}) + \frac{\alpha}{n}K + (1-\alpha)v(\{i\}) \\
&= \frac{v(\{i\})}{K}v(N) + \frac{\alpha}{n}K - \alpha v(\{i\}) \\
&= \varphi^\alpha_i(N, v).
\end{aligned}$$

□

Proof of Corollary 5.1. It is clear that the PD value satisfies efficiency, proportional loss under separatorization, and α-individual rationality for some $\alpha \in [0,1]$. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying the three axioms. From Lemma 5.2, ψ has the form given in (5.3). For any $(N,v) \in \mathcal{G}^N_{nz}$, similar as in the proof of Theorem 5.7, consider a game $(N,w) \in \mathcal{G}^N_{nz}$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$ and $w(N) = \alpha \sum_{j \in N} v(\{j\})$. Since (N,w) is an α-essential game, α-individual rationality implies that $\psi_i(N,w) \geq \alpha w(\{i\})$ for all $i \in N$. By (5.3) applied to (N,w) and (N,w^N), we have $\psi_i(N,w) - \psi_i(N,w^N) = \frac{w(\{i\})}{\sum_{j \in N} w(\{j\})} w(N) - w(\{i\}) = (\alpha - 1)w(\{i\})$. Hence, $\psi_i(N,w^N) = \psi_i(N,w) - (\alpha-1)w(\{i\}) \geq \alpha w(\{i\}) - (\alpha - 1)w(\{i\}) = w(\{i\})$. Efficiency then implies that it must be $\psi_i(N,w^N) = w(\{i\})$ for all $i \in N$, since $w^N(N) = \sum_{j \in N} w(\{j\})$.

Since $(N,v^N) = (N,w^N)$, then $\psi_i(N,v^N) = w(\{i\}) = v(\{i\})$. Again, by (5.3) applied to (N,v) and (N,v^N), we have $\psi_i(N,v) = \frac{v(\{i\})}{K} v(N) - v(\{i\}) + \psi_i(N,v^N) = \frac{v(\{i\})}{K} v(N) - v(\{i\}) + v(\{i\}) = \frac{v(\{i\})}{K} v(N) = PD_i(N,v)$. □

5.8 Conclusion

One of the main issues in economic allocation problems is the trade-off between egalitarianism and egocentrism. The PD value applies an egocentric principle by first assigning to every player its own stand-alone worth, and then allocates the remaining surplus among all players proportional to their stand-alone worths. The EPSD value focuses on egalitarianism in allocating the stand-alone worths by first assigning to every player the average of all stand-alone worths, and also allocates the remaining surplus among all players proportional to their stand-alone worths.

In this chapter, we have introduced the family of proportional division surplus values, being the convex combinations of the EPSD and PD values. These values make a trade-off between egalitarianism and egocentrism. Therefore, this is similar in spirit to the literature that combines diffferent economic allocation principles, such as also, for example, the egalitarian Shapley values, the consensus values, or the convex combinations of the ESD and ED values. We provided characterizations for this family of values as well as any member belonging to this family using two parallel axioms on a fixed player set based on player separatorization. In particular, we showed that weak no advantageous reallocation and proportional loss under separatorization, together with some standard axioms such as efficiency, anonymity and weak linearity, characterized the class of affine combinations of the EPSD and PD values (Theorem 5.2). Additionally adding superadditive monotonicity characterized a subclass of these values (Theorem 5.3) and then replacing anonymity by weak desirability characterizes the class of convex combinations of the EPSD and PD values (Theorem 5.4). We obtained similar results using proportional balanced contributions under separatorization instead of proportional loss under separatorization (Theorem 5.5).

As argued also in, for example, the literature on bankruptcy problems, a disadvantage of proportional division is that players with (relatively) small claims/stand-alone worths, might receive a very low share in the resource to be divided. This can be dampened by first giving all players a uniform fixed share of the resource, and allocating the remainder proportional to the claims/stand-alone worths. For example, in the convex combinations of the EPSD and PD values the players maximally are guaranteed a uniform share in the sum of the stand-alone worths (if this is possible, i.e. when the worth of the grand coalition is at least equal to the sum of the stand-alone worths), with the EPSD value being the extreme where all players first get a uniform share in the sum of the worths of the stand-alone worths. Affine combinations of the EPSD and PD values allow other initial uniform shares.

Finally, using parameterized axioms that depend on the 'weight' α characterized specific values in this class (Theorems 5.6 and 5.7). The study of other characterizations for the family of proportional surplus division values is left for future research. It also seems to be worthwhile to investigate the convex combinations of the EPSD and PD values for claims problems.

Chapter 6

Equal Loss under Separatorization and Egalitarian Values

6.1 Introduction

The equal division (ED) value and the equal surplus division (ESD) value are two well-known egalitarian values for TU-games. In particular, the ED value, the ESD value, and the classes of their affine and convex combinations have been given a number of axiomatic characterizations. In the previous two chapters, we axiomatically compared the two values with proportional values. In this chapter, which is based on Zou and van den Brink (2020), we develop new characterizations of the ED value, the ESD value, and the classes of affine and convex combinations of the ED and ESD values.

Our characterizations involve a new axiom relying on separatorization. As mentioned in Chapter 5, separatorization of a player refers to the complete loss of productive potential of cooperation that the worth of any coalition containing this player equals the sum of the stand-alone worths of the players in this coalition, while the worth of any coalition without her remains unchanged. This operation is in line with 'veto-ification' introduced in van den Brink and Funaki (2009), dummification introduced in Béal et al. (2018), and nullification studied in Gómez-Rúa and Vidal-Puga (2010), Béal et al. (2016b), Ferrières (2017), Kongo (2018), Kongo (2019), and Kongo (2020). The difference among them is which role that a player acts as. Specifically, veto-ification, dummification, nullification, and separatorization, respectively, suppose a player becoming a veto player, a dummy player, a null player, and a separator (also known as a dummifying player in Casajus and Huettner (2014a)) in a TU-game. There exist several axioms which evaluate the consequences of the aforementioned operations in TU-games. Assuming the same change in payoff for all other players under such operation, van den Brink and Funaki (2009) suggest the veto equal loss property for the ED value, and Ferrières (2017) and Kongo (2018) independently suggest the nullified equal loss property for the ED value, the ESD value and the classes of their affine and convex combinations. Similarly, we define the axiom of equal loss under separatorization imposing the same requirement, except that a player becomes a separator.

In this chapter, we show that equal loss under separatorization and efficiency yield a family of values that all have in common that they equally split the worth of the grand coalition. This family is not identical to the family implied by the axioms of the nullified equal loss property and efficiency as given by Ferrières (2017). We characterize the class of affine combinations of the ED and ESD values by using the two axioms in addition to fairness (van den Brink, 2002) and homogeneity. While Ferrières (2017) characterizes the classes of affine and convex combinations of the ED and ESD values involving the nullified equal loss property, we highlight that replacing the nullified equal loss property by equal loss under separatorization yields a new characterization. Moreover, parallel to the axiomatic results in Kongo (2018), we provide characterizations of both the ED value and the ESD value. Besides, we provide alternative characterizations of the classes of affine and convex combinations of the ED and ESD values, which are similar to that of the PD and EPSD values in Subsection 5.4.1.

This chapter is organized as follows. Section 6.2 introduces the concept of equal loss under separatorization. Section 6.3 and Section 6.4 present the main results. Section 6.5 shows the logical independence of the axioms in the characterization results. All proofs are provided in Section 6.6. Section 6.7 concludes.

6.2 Equal loss under separatorization

Before stating the characterizations, we briefly recall the definition of separatorization (see Section 5.2 in details, but the domain $\mathcal{G}^N_{nz}$ is replaced by the class of all games on player set N, i.e. $\mathcal{G}^N$) and then introduce a new axiom called equal loss under separatorization.

Given a TU-game, as in Section 5.2, separatorization of a player means that the worth of any coalition containing this player becomes equal to the sum of the stand-alone worths of the players in this coalition. Formally, for $(N, v) \in \mathcal{G}^N$ and $h \in N$, we denote by (N, v^h) the TU-game from (N, v) if player h becomes a separator, i.e.

$$v^h(S) = \begin{cases} \sum_{j \in S} v(\{j\}) & \text{if } S \subseteq N, h \in S, \\ v(S) & \text{otherwise.} \end{cases}$$

As argued in Chapter 5, $(v^i)^j = (v^j)^i$ for every pair $i, j \in N$, and thus for every coalition $S \subseteq N$, (N, v^S), where the players in S became separators, is well-defined and does not depend on the order in which the players become separators. Formally, $v^S(T) = \sum_{j \in T} v(\{j\})$ if $T \cap S \neq \emptyset$, and $v^S(T) = v(T)$ otherwise, is obtained by sequentially separatizing the players in S in any order. Note that (N, v^N) is the corresponding additive TU-game of (N, v), namely $v^N(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$.

The following new axiom imposes that if a player becomes a separator, then all other players should be affected equally.

- **Equal loss under separatorization**. For all $(N, v) \in \mathcal{G}^N$, all $h \in N$ and all $i, j \in N \backslash \{h\}$,

$$\psi_i(N, v) - \psi_i(N, v^h) = \psi_j(N, v) - \psi_j(N, v^h). \tag{6.1}$$

Clearly, this axiom is closely related to proportional loss under separatorization as discussed in Subsection 5.4.1, and is considered here on the class of TU-games $\mathcal{G}^N$ with $|N| \geq 3$.

6.3 Axiomatic characterizations

In this section, we characterize the ED value, the ESD value, and the classes of their affine and convex combinations on the class of TU-games with at least three players.

6.3.1 Axiomatizations of the family of affine combinations of ED and ESD

Before stating the characterizations, we derive a useful property implied by the combination of efficiency and equal loss under separatorization.

Lemma 6.1. *Let $|N| \geq 3$. If a value ψ on $\mathcal{G}^N$ satisfies efficiency and equal loss under separatorization, then*

$$\psi_i(N, v) - \psi_i(N, v^N) = \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})], \tag{6.2}$$

for all $(N, v) \in \mathcal{G}^N$ and $i \in N$.

The proof of Lemma 6.1 and of all other results in this chapter can be found in Section 6.6.

Lemma 6.1 is similar to Lemma 5.2, but considers, for efficient values, the consequences of equal loss under separatorization instead of proportional loss under separatorization.

Remark 6.1. Lemma 6.1 indicates that any value on $\mathcal{G}^N$ satisfying efficiency and equal loss under separatorization is uniquely determined by an efficient value determined on additive TU-games since $v^N(S) = \sum_{j \in S} v(\{j\})$ for all $(N, v) \in \mathcal{G}^N$ and $S \subseteq N$. This means that, efficiency and equal loss under separatorization in addition to some axiom(s) that determine the payoff allocation for additive TU-games, characterize a unique value on $\mathcal{G}^N$.

Remark 6.2. Any value with the form of (6.2) satisfies equal loss under separatorization, but need not satisfy efficiency. For instance, the value $\psi = ED + a$, where $a \in \mathbb{R}^N$ is such that $\sum_{j \in N} a_i \neq 0$, also satisfies (6.2) but not efficiency.

To characterize the class of affine combinations of the ED and ESD values, we recall the well-known axioms of fairness and homogeneity.

- **Fairness** (van den Brink, 2002). For all $(N, v), (N, w) \in \mathcal{G}^N$ and all $i, j \in N$ such that i and j are symmetric in (N, w), it holds that $\psi_i(N, v + w) - \psi_i(N, v) = \psi_j(N, v + w) - \psi_j(N, v)$.

- **Homogeneity**. For all $(N, v) \in \mathcal{G}^N$ and all $c \in \mathbb{R}$, it holds that $\psi(N, cv) = c\psi(N, v)$.

Theorem 6.1. *A value ψ on $\mathcal{G}^N$ satisfies efficiency, equal loss under separatorization, fairness, and homogeneity if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1 - \beta)ED$.*

Note that linearity implies homogeneity, and linearity and symmetry together imply fairness. The following corollary is a direct consequence of Theorem 6.1. For completeness, its proof is also given in Section 6.6.

Corollary 6.1. Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N$ satisfies efficiency, equal loss under separatorization, linearity, and symmetry if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1 - \beta)ED$.

Remark 6.3. As mentioned in Chapter 1, under efficiency, linearity, and symmetry, the nullifying player property characterizes the ED value in van den Brink (2007), and the dummifying player property characterizes the ESD value in Casajus and Huettner (2014a). Therefore, the difference among the ED value, the ESD value, and the class of their affine combinations is pinpointed to one axiom. Mind that, from Remark 6.1, under efficiency and equal loss under separatorization, the dummifying player property characterizes the ESD value, whereas the nullifying player property cannot characterize the ED value. Consider, for example, $\psi_i(N, v) = \frac{v(N)}{n} + a_i(N, v)v(\{i\})$ for all $i \in N$, where $a\colon \mathcal{G}^N \to \mathbb{R}^N$ is a function such that (i) $a(N, v) = a(N, w)$ of $v(\{i\}) = w(\{i\})$ for all $i \in N$, and (ii) $\sum_{i \in N} a_i(N, v)v(\{i\}) = 0$ for all $(N, v) \in \mathcal{G}^N$. This value also satisfies the three axioms.

In the next result, we present a new characterization of the class of convex combinations of the ED and ESD values.

Theorem 6.2. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N$ satisfies efficiency, equal loss under separatorization, additivity, desirability, and superadditive monotonicity if and only if there is $\beta \in [0, 1]$ such that $\psi = \beta ESD + (1 - \beta)ED$.*

Corollary 6.1 and Theorem 6.2 show that Theorem 1.11 and Theorem 1.12 (see Ferrières (2017)) are still valid if the nullified equal loss property is replaced by equal loss under separatorization, although (6.2) does not coincide with the formula of values satisfying efficiency and the nullified equal loss property (see Formula (3), Ferrières, 2017).

6.3.2 Axiomatizations of the ED value and the ESD value

Notice that nullification of all players in a TU-game leads to the null game, whereas separatorization of all players leads to the corresponding additive game. The null game property requires that all players gain zero for any null game. This axiom is well adapted to the representation of a special allocation among players under nullification, but not separatorization. Thus, the null game property is used in Theorem 1.13, as well as other axiomatic results in Kongo (2018) and Kongo (2019). Interestingly, Theorem 1.13 is still valid when we use equal loss under separatorization instead of the nullified equal loss property. To show this, we first characterize the ED value using the axiom of nonnegativity.

Lemma 6.2. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}^N$ satisfies efficiency, equal loss under separatorization, and nonnegativity if and only if $\psi = ED$.*

Next, we use equal loss under separatorization in combination with monotonicity axioms to characterize the ED, respectively ESD value.

Theorem 6.3. *Let $|N| \geq 3$. Let ψ be a value on $\mathcal{G}^N$ that satisfies efficiency, equal loss under separatorization, and the null game property. Then,*

(i) *ψ satisfies grand coalition monotonicity if and only if $\psi = ED$.*

(ii) *ψ satisfies Id+sur monotonicity if and only if $\psi = ESD$.*

6.4 Alternative axiomatizations using homogeneity

Equal loss under separatorization and proportional loss under separatorization respectively suppose equality and proportionality on allocation rules when a player becomes a separator. In Subsection 5.4.1, the axiomatic results on the classes of affine and convex combinations of the PD and EPSD values use proportional loss under separatorization. As a contrast, this subsection provides axiomatizations of the classes of affine and convex combinations of the ED and ESD values.

We will employ the axioms of anonymity, weak no advantageous reallocation, continuity, and weak desirability, which are described in Subsection 1.3 and Subsection 5.2, but here they are considered on the class $\mathcal{QA}^N$.

Without going into details, we first provide Theorem 6.4 that characterizes a family of values on the domain of quasi-additive games, of which the proof is similar to that of Theorem 5.1. This theorem is also a direct extension of Theorem 1 in Chun (1988), and the proof is omitted.

Theorem 6.4. *Let $|N| \geq 3$. A value ψ on $\mathcal{QA}^N$ satisfies efficiency, anonymity, weak no advantageous reallocation, and continuity if and only if there exists a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\psi_i(N, v) = \frac{v(N)}{n} + \left(v(\{i\}) - \frac{\sum_{j \in N} v(\{j\})}{n}\right) g\Big(\sum_{j \in N} v(\{j\}), v(N)\Big) \qquad (6.3)$$

for all $(N, v) \in \mathcal{QA}^N$ *and* $i \in N$.

Remark 6.4. The axioms invoked in Theorem 5.1 and Theorem 6.4 are the same except that they are respectively considered on $\mathcal{QA}^N_{nz}$ and $\mathcal{QA}^N$. For every $(N, v) \in \mathcal{QA}^N_{nz}$, since $\sum_{j\in N} v(\{j\}) \neq 0$, then $g' : \mathbb{R}\backslash\{0\} \times \mathbb{R} \to \mathbb{R}$ is well-defined by

$$g(\sum_{j\in N} v(\{j\}), v(N)) = \frac{v(N)}{\sum\limits_{j\in N} v(\{j\})} - \frac{1}{\sum\limits_{j\in N} v(\{j\})} g'(\sum_{j\in N} v(\{j\}), v(N)).$$

Then, (6.3) can be written as

$$\psi_i(N, v) = \frac{v(\{i\})v(N)}{\sum_{j\in N} v(\{j\})} - \left(\frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} - \frac{1}{n}\right) g'(\sum_{j\in N} v(\{j\}), v(N)),$$

which coincides with (5.4).

Remark 6.5. Similar to Remark 5.3, if continuity is replaced by continuity in least at one point, then it does not affect (6.3), but only affects that g is no longer required to be continuous.

Theorem 6.5. *Let* $|N| \geq 3$. *A value* ψ *on* $\mathcal{G}^N$ *satisfies efficiency, anonymity, weak no advantageous reallocation, equal loss under separatorization, and homogeneity if and only if there is* $\beta \in \mathbb{R}$ *such that* $\psi = \beta ESD + (1 - \beta)ED$.

The next theorem provides an alternative axiomatization of the family of convex combinations of the ED and ESD values. This theorem is similar to Theorem 5.4, but the proof uses Lemma 6.3 instead of Lemma 5.3.

Theorem 6.6. *Let* $|N| \geq 3$. *A value* ψ *on* $\mathcal{G}^N$ *satisfies efficiency, weak no advantageous reallocation, equal loss under separatorization, homogeneity, superadditive monotonicity, and weak desirability if and only if there is* $\beta \in [0, 1]$ *such that* $\psi = \beta ESD + (1 - \beta)ED$.

Lemma 6.3 reveals that weak desirability together with some of the axioms in Theorem 6.5 imply anonymity.

Lemma 6.3. *On* $\mathcal{G}^N$ *with* $|N| \geq 3$, *efficiency, weak no advantageous reallocation, equal loss under separatorization, and weak desirability imply anonymity.*

Theorem 6.5 and Theorem 6.6 are still valid if homogeneity is replaced by weak linearity on $\mathcal{G}^N$, which are similar to Theorem 5.2 and Theorem 5.4.

- **Weak linearity**. For all $(N, v), (N, w) \in \mathcal{G}^N$ and $a \in \mathbb{R}$ such that there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, it holds that $\psi(N, av + w) = a\psi(N, v) + \psi(N, w)$.

To conclude, in Table 6.1 the axiomatic results (except Theorem 6.1) stated in this chapter are summarized. In this table, '$\surd$' has the meaning that the properties are used in each theorem.

TABLE 6.1: Properties of the values for TU-games

Values	$\beta ESD + (1-\beta)ED$					
	$\beta \in \mathbb{R}$		$\beta \in [0,1]$		$\beta = 0$	$\beta = 1$
Efficiency	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Linearity	$\surd$					
Additivity			$\surd$			
Homogeneity		$\surd$		$\surd$		
Symmetry	$\surd$					
Anonymity		$\surd$				
Equal loss under separatorization	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
The null game property					$\surd$	$\surd$
Superadditive monotonicity			$\surd$	$\surd$		
Desirability			$\surd$			
Weak desirability				$\surd$		
Grand coalition monotonicity					$\surd$	
Id+sur monotonicity						$\surd$
Weak no advantageous reallocation		$\surd$		$\surd$		
	Th.6.1	Th.6.5	Th.6.2	Th.6.6	Th.6.3(i)	Th.6.3(ii)

6.5 Independence of axioms

Logical independence of the axioms used in the characterization results can be shown by the following alternative values.

Theorem 6.1:

(i) The value defined for all $(N, v) \in \mathcal{G}^N$ and $i \in N$ by

$$\psi_i(N, v) = 0 \tag{6.4}$$

satisfies all axioms except efficiency.

(ii) The Shapley value satisfies all axioms except equal loss under separatorization.

(iii) The value defined for all $(N, v) \in \mathcal{G}^N$ with $N = \{1, 2, ..., n\}$ and $i \in N$ by

$$\psi_i(N, v) = \frac{i}{\sum_{j \in N} j} \sum_{j \in N} v(\{j\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})] \tag{6.5}$$

satisfies all axioms except fairness.

(iv) Let $a \in \mathbb{R}^N$ be such that $\sum_{i \in N} a_i = 0$ and $a \neq 0$. The value defined for all $(N, v) \in \mathcal{G}^N$ and $i \in N$ by

$$\psi_i(N, v) = \frac{v(N)}{n} + a_i \tag{6.6}$$

satisfies all axioms except homogeneity.

Corollary 6.1:

(i) The value defined by (6.4) satisfies all axioms except efficiency.

(ii) The Shapley value satisfies all axioms except equal loss under separatorization.

(iii) The value defined by (6.5) satisfies all axioms except symmetry.

(iv) The value defined for all $(N,v) \in \mathcal{G}^N$ by

$$\psi(N,v) = \begin{cases} ED(N,v) & \text{if } v(S) > 0 \text{ for all } S \subseteq N \text{ with } |S| = 1; \\ ESD(N,v) & \text{otherwise.} \end{cases} \tag{6.7}$$

satisfies all axioms except linearity.

Theorem 6.2:

(i) The value defined by (6.4) satisfies all axioms except efficiency.

(ii) The value defined for all $(N,v) \in \mathcal{G}^N$ with $N = \{1,2,...,n\}$ by $\psi_1(N,v) = v(N)$ and $\psi_i(N,v) = 0$ for any $i \neq 1$ satisfies all axioms except equal loss under separatorization.

(iii) The value defined by (6.7) satisfies all axioms except additivity.

(iv) The value defined for all $(N,v) \in \mathcal{G}^N$ by $\psi(N,v) = 2ED(N,v) - ESD(N,v)$ satisfies all axioms except desirability.

(v) The value defined for all $(N,v) \in \mathcal{G}^N$ by $\psi(N,v) = 2ESD(N,v) - ED(N,v)$ satisfies all axioms except superadditive monotonicity.

Theorem 6.3:

(i) The value defined by (6.4) satisfies all axioms except efficiency.

(ii) The value defined for all $(N,v) \in \mathcal{G}^N$ with $N = \{1,2,...,n\}$ by $\psi_i(N,v) = \frac{i}{\sum_{j\in N} j} v(N)$ for all $i \in N$ satisfies all axioms of Theorem 6.3(i) except equal loss under separatorization.

(iii) The value defined by (6.6) satisfies all axioms of Theorem 6.3(i) except the null game property.

(iv) The value defined for all $(N,v) \in \mathcal{G}^N$ with $N = \{1,2,...,n\}$ and $i \in N$ by

$$\psi_i(N,v) = v(\{i\}) + \frac{i}{\sum_{j\in N} j}[v(N) - \sum_{j\in N} v(\{j\})]$$

satisfies all axioms of Theorem 6.3(ii) except equal loss under separatorization.

(v) Let $a \in \mathbb{R}^N$ be such that $\sum_{j\in N} a_j = 0$ and $a \neq 0$. The value defined for all $(N,v) \in \mathcal{G}^N$ by $\psi(N,v) = ESD(N,v) + a$ satisfies all axioms of Theorem 6.3(ii) except the null game property.

(vi) The value defined by (6.5) satisfies all axioms, but neither grand coalition monotonicity nor Id+sur monotonicity.

Theorems 6.5 and 6.6 :

(i) The value defined by (6.4) satisfies all axioms except efficiency.

(ii) The value defined for all $(N,v) \in \mathcal{G}^N$ and all $i \in N$, by

$$\psi_i(N,v) = \begin{cases} \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})] + \frac{(v(\{i\}))^2}{\sum_{j\in N}(v(\{j\}))^2} \sum_{j\in N} v(\{j\}), & \text{if } (N,v) \in \mathcal{G}^N_{nz}, \\ \frac{v(N)}{n}, & \text{otherwise.} \end{cases}$$

satisfies all axioms except weak no advantageous reallocation.

(iii) The value defined by (6.5) satisfies all axioms except anonymity and weak desirability.

(iv) The Shapley value satisfies all axioms except equal loss under separatorization.

(v) The value defined for all $(N,v) \in \mathcal{G}^N$ by

$$\psi(N,v) = \begin{cases} ESD(N,v), & \text{if } \sum_{j\in N} v(\{j\}) \geq 0, \\ ED(N,v), & \text{otherwise.} \end{cases}$$

satisfies all axioms except homogeneity.

(vi) The value defined for all $(N,v) \in \mathcal{G}^N$ by $\psi(N,v) = 2ESD(N,v) - ED(N,v)$ satisfies all axioms of Theorem 6.6 except superadditive monotonicity.

6.6 Proofs

Proof of Lemma 6.1. Let ψ be a value on $\mathcal{G}^N$, $|N| \geq 3$, satisfying efficiency and equal loss under separatorization. We divide the proof in three steps.

Step 1. By equal loss under separatorization, (6.1) is satisfied for any triple of players. Taking $h \in N$ and $i \in N\backslash\{h\}$, summing (6.1) over $j \in N\backslash\{h\}$ and using efficiency yields that for all $(N,v) \in \mathcal{G}^N$, $h \in N$ and $i \in N\backslash\{h\}$,

$$\begin{aligned} \psi_i(N,v) - \psi_i(N,v^h) &= \frac{1}{n-1}\left[\sum_{j\in N\backslash\{h\}} \psi_j(N,v) - \sum_{j\in N\backslash\{h\}} \psi_j(N,v^h)\right] \\ &= \frac{1}{n-1}[v(N) - \psi_h(N,v) - v^h(N) + \psi_h(N,v^h)]. \end{aligned} \tag{6.8}$$

Step 2. Next, we show that for all $(N, v) \in \mathcal{G}^N$ and $S \subseteq N$ with $1 \leq |S| \leq n-1$,

$$\psi(N, v^S) = \psi(N, v^N). \tag{6.9}$$

We derive the assertion by an induction on the number of separators.

Initialization. Since $(N, v^{N\setminus\{h\}}) = (N, v^N)$ for any $h \in N$, then $\psi(N, v^S) = \psi(N, v^N)$ for all $S \subseteq N$ with $|S| = n-1$,

Induction hypothesis (IH). Assume that $\psi(N, v^T) = \psi(N, v^N)$ holds for all $T \subseteq N$ with $|T| = t$, $2 \leq t \leq n-1$.

Induction step. Consider $(N, v^S) \in \mathcal{G}^N$ and $S \subset N$ such that $|S| = t-1$. Since $v^S(N) = \sum_{k \in N} v(\{k\})$ and $v^S(\{k\}) = v(\{k\})$ for all $k \in N$, then by (6.8) applied to (N, v^S), we obtain, using $v^S(N) = v^{S\cup\{h\}}(N)$, that for all $i \neq h$,

$$\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{h\}}) = \frac{1}{n-1}[-\psi_h(N, v^S) + \psi_h(N, v^{S\cup\{h\}})]. \tag{6.10}$$

Pick any $j \in N\setminus S$ and $i \in N\setminus(S \cup \{j\})$ (which is possible since $|S| \leq n-2$). We obtain

$$\begin{aligned}
\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}}) &= \frac{1}{n-1}[-\psi_j(N, v^S) + \psi_j(N, v^{S\cup\{j\}})] \\
&= \frac{1}{n-1}[-\psi_j(N, v^S) + \psi_j(N, v^N)] \\
&= \frac{1}{n-1}[-\psi_j(N, v^S) + \psi_j(N, v^{S\cup\{i\}})] \\
&= \frac{1}{n-1}\Big[\frac{1}{n-1}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{i\}})]\Big] \\
&= \frac{1}{(n-1)^2}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}})],
\end{aligned}$$

where the first and fourth equalities hold from (6.10), and the other three equalities hold by the induction hypothesis.

Since $\frac{n^2-2n}{(n-1)^2} \neq 0$, this implies that $\psi_i(N, v^S) = \psi_i(N, v^{S\cup\{j\}})$ for every $i \in N\setminus(S \cup \{j\})$. Pick any $k \in S$. By equal loss under separatorization, we have $\psi_k(N, v^S) - \psi_k(N, v^{S\cup\{j\}}) = \psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}}) = 0$, which implies

$$\psi_k(N, v^S) = \psi_k(N, v^{S\cup\{j\}}).$$

Since $v^S(N) = v^{S\cup\{j\}}(N)$, efficiency then implies that $\psi_j(N, v^S) = \psi_j(N, v^{S\cup\{j\}})$. There exists such $j \in N$ for each $S \subsetneq N$, so that

$$\psi(N, v^S) = \psi(N, v^{S\cup\{j\}}) = \psi(N, v^N),$$

where the latter equality follows from the induction hypothesis.

Step 3. By (6.9), $\psi(N,v^h) = \psi(N,v^N)$ for all $h \in N$. Then (6.8) implies that for two distinct players $i,h \in N$,

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{1}{n-1}[v(N) - \psi_h(N,v) - v^N(N) + \psi_h(N,v^N)].$$

Summing the above equality over $h \in N\backslash\{i\}$ yields

$$\begin{aligned}
&(n-1)[\psi_i(N,v) - \psi_i(N,v^N)] \\
&= \frac{1}{n-1}\left[(n-1)[v(N) - v^N(N)] - \sum_{h\in N\backslash\{i\}}(\psi_h(N,v) - \psi_h(N,v^N))\right] \\
&= \frac{1}{n-1}[(n-2)[v(N) - v^N(N)] + [\psi_i(N,v) - \psi_i(N,v^N)]],
\end{aligned}$$

where the second equality follows from efficiency. It follows that $\frac{n(n-2)}{n-1}[\psi_i(N,v) - \psi_i(N,v^N)] = \frac{n-2}{n-1}[v(N) - v^N(N)]$, which implies (6.2) since $\frac{n-2}{n-1} \neq 0$ (by $n \geq 3$). □

Proof of Theorem 6.1. Existence is obvious. For the uniqueness part, let ψ be a value on $\mathcal{G}^N$ that satisfies the four axioms. By Lemma 6.1 and Remark 6.1, we have to show that $\psi(N,v) = \beta ESD(N,v) + (1-\beta)ED(N,v)$, $\beta \in \mathbb{R}$, for all additive games (N,v). Let $D(N,v) = \{i \in N \mid v(\{i\}) \neq 0\}$. We prove uniqueness by induction on $d(N,v) = |D(N,v)|$.

Initialization. If $d(N,v^0) = 0$, i.e. (N,v^0) is the null game, then homogeneity implies that $\psi_i(N,v^0) = 0$ for all $i \in N$.

Suppose that $d(N,v) = 1$, i.e. $v = v(\{i\})u_{\{i\}}$. Since any $j,k \in N\backslash\{i\}$ are symmetric in $(N,u_{\{i\}})$, fairness implies that $\psi_j(N,v^0 + u_{\{i\}}) - \psi_j(N,v^0) = \psi_k(N,v^0 + u_{\{i\}}) - \psi_k(N,v^0)$, and thus $\psi_j(N,u_{\{i\}}) = \psi_k(N,u_{\{i\}})$. Efficiency then implies that for all $j \in N\backslash\{i\}$,

$$\psi_j(N,u_{\{i\}}) = \frac{1-\psi_i(N,u_{\{i\}})}{n-1}. \tag{6.11}$$

Next, pick any $i,j \in N$ with $i \neq j$, and consider $(N,-u_{\{i\}})$ and $(N,u_{\{i\}} + u_{\{j\}})$. Since i and j are symmetric in $(N,u_{\{i\}} + u_{\{j\}})$, fairness implies that

$$\psi_i(N,-u_{\{i\}} + u_{\{i\}} + u_{\{j\}}) - \psi_i(N,-u_{\{i\}}) = \psi_j(N,-u_{\{i\}} + u_{\{i\}} + u_{\{j\}}) - \psi_j(N,-u_{\{i\}}).$$

By homogeneity,

$$\psi_i(N,u_{\{j\}}) + \psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}) + \psi_j(N,u_{\{i\}}). \tag{6.12}$$

Combining (6.11) with (6.12) yields

$$\frac{1-\psi_j(N,u_{\{j\}})}{n-1} + \psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}) + \frac{1-\psi_i(N,u_{\{i\}})}{n-1}.$$

Since

$$\psi_i(N,u_{\{i\}}) - \frac{1-\psi_i(N,u_{\{i\}})}{n-1} = \frac{(n-1)\psi_i(N,u_{\{i\}}) - 1 + \psi_i(N,u_{\{i\}})}{n-1}$$
$$= \frac{n.\psi_i(N,u_{\{i\}}) - 1}{n-1}$$

and similar for j, it follows that

$$\psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}). \tag{6.13}$$

According to (6.13), setting $a = \psi_i(N,u_{\{i\}})$ for all $i \in N$, and $\beta = \frac{na-1}{n-1}$, for $v = v(\{i\})u_{\{i\}}$, we have

$$\begin{aligned}
&\beta\, ESD_i(N,v) + (1-\beta)ED_i(N,v) \\
&= \frac{na-1}{n-1}\left(v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(\{j\})\right)\right) + \frac{n(1-a)}{n-1}\cdot\frac{v(N)}{n} \\
&= \frac{na-1}{n-1}v(\{i\}) + 0 + \frac{1-a}{n-1}v(\{i\}) \\
&= v(\{i\})a \\
&= v(\{i\}\psi_i(N,u_{\{i\}}) \\
&= \psi_i(N,v),
\end{aligned}$$

where the last equality follows from homogeneity.

By (6.11) and homogeneity,

$$\psi_j(N,v) = \frac{1-a}{n-1}v(\{i\}) = \beta\, ESD_j(N,v) + (1-\beta)ED_j(N,v)$$

for all $j \in N\backslash\{i\}$.

Induction hypothesis. Assume that $\psi(N,v')$ is uniquely determined whenever $d(N,v') = k, 1 \leq k \leq n-1$.

Induction step. Let $(N,v) \in \mathcal{G}^N$ be an additive game such that $d(N,v) = k+1$. Take $h \in D(N,v)$, and consider game (N,v') given by $v' = v - v(\{h\})u_{\{h\}}$. Take a $j \in N\backslash\{h\}$. Then, for all $i \in N\backslash\{j,h\}$, fairness implies that

$$\psi_i(N,v) - \psi_j(N,v) = \psi_i(N,v') - \psi_j(N,v'), \tag{6.14}$$

where the right-hand side is determined by the induction hypothesis.

Take $g \in D(N,v)\backslash\{h\}$ (which exists since $d(N,v) \geq 2$) and $j \in N\backslash\{g,h\}$ (which exists since $n \geq 3$), and consider $v'' = v - v(\{g\})u_{\{g\}}$. Then fairness implies

$$\psi_h(N,v) - \psi_j(N,v) = \psi_h(N,v'') - \psi_j(N,v''), \tag{6.15}$$

where the right-hand side is determined by the induction hypothesis.

Finally, efficiency implies that

$$\sum_{i \in N} \psi_i(N, v) = v(N). \tag{6.16}$$

Since the $(n-2)+1+1 = n$ equations (6.14), (6.15) and (6.16) are linearly independent in the n unkown payoffs $\psi_i(N, v)$, these payoffs are uniquely determined.

Thus, the payoffs in any additive game $(N, v) \in \mathcal{G}^N$ are uniquely determined for any choice of $a = \psi_i(N, u_{\{i\}})$, $i \in N$, and thus for any choice of β. Since the corresponding affine combination of the ESD and ED values satisfies the axioms, it must be that $\psi = \beta ESD + (1-\beta)ED$. □

Proof of Corollary 6.1. Existence is obvious. For the uniqueness part, let ψ be a value that satisfies the four axioms. By Lemma 6.1, ψ has the form of (6.2).

First, pick any $i \in N$ and $j \in N \backslash \{i\}$, and consider $(N, u_{\{i\}})$, we have

$$1 - \psi_i(N, u_{\{i\}}) = \sum_{k \in N \backslash \{i\}} \psi_k(N, u_{\{i\}}) = (n-1)\psi_j(N, u_{\{i\}}),$$

where the first equality follows from efficiency, and the second equality follows from symmetry.

Thus,

$$\psi_j(N, u_{\{i\}}) = \frac{1 - \psi_i(N, u_{\{i\}})}{n-1}. \tag{6.17}$$

Second, pick any $i, j \in N$ with $i \neq j$, and consider $(N, u_{\{i\}} + u_{\{j\}})$. Since players i and j are symmetric, symmetry implies that $\psi_i(N, u_{\{i\}} + u_{\{j\}}) = \psi_j(N, u_{\{i\}} + u_{\{j\}})$. By linearity,

$$\psi_i(N, u_{\{i\}}) + \psi_i(N, u_{\{j\}}) = \psi_j(N, u_{\{i\}}) + \psi_j(N, u_{\{j\}}). \tag{6.18}$$

Together with (6.17) and (6.18), we obtain

$$\psi_i(N, u_{\{i\}}) + \frac{1 - \psi_j(N, u_{\{j\}})}{n-1} = \frac{1 - \psi_i(N, u_{\{i\}})}{n-1} + \psi_j(N, u_{\{j\}}).$$

It follows that $\frac{n}{n-1}\psi_i(N, u_{\{i\}}) = \frac{n}{n-1}\psi_j(N, u_{\{j\}})$, which implies

$$\psi_i(N, u_{\{i\}}) = \psi_j(N, u_{\{j\}}). \tag{6.19}$$

Finally, with (6.19), let us set $a = \psi_i(N, u_{\{i\}})$ for all $i \in N$. Obviously, (6.17) implies that $\psi_i(N, u_{\{j\}}) = \frac{1-a}{n-1}$ for all $i, j \in N$ with $i \neq j$. Then, for any $i \in N$,

$$\psi_i(N, v^N) = \psi_i(N, \sum_{j \in N} v(\{j\}) u_{\{j\}})$$

$$
\begin{aligned}
&= \sum_{j\in N} v(\{j\})\psi_i(N, u_{\{j\}}) \\
&= av(\{i\}) + \frac{1-a}{n-1}\sum_{j\in N\setminus\{i\}} v(\{j\}) \\
&= \frac{na-1}{n-1}v(\{i\}) + \frac{1-a}{n-1}\sum_{j\in N} v(\{j\}),
\end{aligned}
$$

where the second equality follows from linearity.

Setting $\beta = \frac{na-1}{n-1}$, then $\psi_i(N, v^N) = \beta v(\{i\}) + \frac{1-\beta}{n}\sum_{j\in N} v(\{j\})$. Plugging this equation into (6.2) yields $\psi = \beta ESD + (1-\beta)ED$. □

Proof of Theorem 6.2. It is clear that $\psi = \beta ESD + (1-\beta)ED$ satisfies the five axioms. Conversely, let ψ be a value on $\mathcal{G}^N$ that satisfies the five axioms. Lemma 5 in Casajus and Huettner (2013) shows that efficiency, additivity and desirability imply linearity. Moreover, desirability implies symmetry. From Corollary 6.1, there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1-\beta)ED$. Furthermore, desirability brings $\beta \geq 0$, and superadditive monotonicity brings $\beta \leq 1$. □

Proof of Lemma 6.2. It is clear that the ED value satisfies efficiency, equal loss under separatorization, and nonnegativity. Conversely, suppose that ψ is a value on $\mathcal{G}^N$ that satisfies the three axioms. For any $(N, v) \in \mathcal{G}^N$, consider $(N, w) \in \mathcal{G}^N$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$ and $w(N) = 0$. By (6.2) (see Lemma 6.1) applied to (N, w) and (N, w^N), we have $\psi_i(N, w) - \psi_i(N, w^N) = -\frac{1}{n}\sum_{j\in N} w(\{j\})$ for all $i \in N$. It follows that

$$
\psi_i(N, w^N) = \psi_i(N, w) + \frac{1}{n}\sum_{j\in N} w(\{j\}) \geq \frac{1}{n}\sum_{j\in N} w(\{j\}),
$$

where the last inequality holds from nonnegativity. Then, efficiency implies that $\psi_i(N, w^N) = \frac{1}{n}\sum_{j\in N} w(\{j\})$ for all $i \in N$.

Since $(N, v^N) = (N, w^N)$, then $\psi_i(N, v^N) = \frac{1}{n}\sum_{j\in N} v(\{j\})$. Again, by (6.2) but now applied to (N, v) and (N, v^N), we have

$$
\begin{aligned}
\psi_i(N, v) &= \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})] + \psi_i(N, v^N) \\
&= \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})] + \frac{1}{n}\sum_{j\in N} v(\{j\}) \\
&= \frac{1}{n}v(N),
\end{aligned}
$$

as desired. □

Proof of Theorem 6.3. (i) Existence is obvious. Uniqueness follows from Lemma 6.2 and the fact that the null game property and grand coalition monotonicity imply nonnegativity.

(ii) Existence is obvious. For the uniqueness part, let ψ be a value that satisfies the four axioms. Consider two additive games $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(\{i\}) = w(\{i\})$. By Id+sur monotonicity, $\psi_i(N, v) = \psi_i(N, w)$, which means that i's payoff depends only on her stand-alone worth. Next, consider the additive game $(N, v') \in \mathcal{G}^N$ such that $v'(\{i\}) = v(\{i\})$ and $v'(\{j\}) = 0$ for all $j \in N \backslash \{i\}$, and let $(N, v^0) \in \mathcal{G}^N$ be the null game. It holds that $\psi_i(N, v) = \psi_i(N, v') = v(\{i\}) - \sum_{j \in N \backslash \{i\}} \psi_j(N, v') = v(\{i\}) - \sum_{j \in N \backslash \{i\}} \psi_j(N, v^0) = v(\{i\})$, where the second equality follows from efficiency, and the last equality follows from the null game property. The assertion immediately follows from Remark 6.1. □

Proof of Theorem 6.5. The proof is similar to that of Theorem 5.2, but we put it here for completeness as follows.

It is clear that $\psi = \beta ESD + (1 - \beta)ED$, $\beta \in \mathbb{R}$, satisfies the five axioms. Conversely, let ψ be a value on $\mathcal{G}^N$ that satisfies the five axioms. The proof is divided into two steps. In Step 1, we derive the formula of ψ on $\mathcal{QA}^N$; in Step 2, we extend the formula obtained in Step 1 from $\mathcal{QA}^N$ to $\mathcal{G}^N$.

Step 1. Consider any game $(N, v) \in \mathcal{QA}^N$ and $(N, v^N) \in \mathcal{A}^N$. From Lemma 6.1,

$$\psi_i(N, v) - \psi_i(N, v^N) = \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})] \text{ for all } i \in N. \tag{6.20}$$

Since (N, v^N) is an additive game, it must be that $\psi_i(N, v^N)$ doesn't have the terms of $v(S)$, $S \subseteq N$, $|S| \neq 1$. Considering that the right-hand side of (6.20) only has the terms of $v(S)$ with $|S| = 1, n$, we obtain from (6.20) that $\psi_i(N, v)$ has the term $\frac{1}{n}v(N)$, but no terms of $v(S)$, $S \subseteq N$, $1 < |S| < n$. This implies that $\psi_i(N, v)$ is a continuous function with respect to $v(S)$, $S \subseteq N$, $|S| \neq 1$.

As stated in Theorem 6.4 and Remark 6.5, $\psi_i(N, v)$ and $\psi_i(N, v^N)$ have the form of (6.3). Substituting them into (6.20), we obtain for every $i \in N$,

$$\begin{aligned}
&\psi_i(N, v) - \psi_i(N, v^N) - \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})] \\
&= \left(v(\{i\}) - \frac{\sum_{j \in N} v(\{j\})}{n}\right)(g(\sum_{j \in N} v(\{j\}), v(N)) - g(\sum_{j \in N} v(\{j\}), v^N(N)) \\
&= 0,
\end{aligned} \tag{6.21}$$

where the last equality follows from (6.20).

To obtain the formula of $\psi_i(N, v)$, $(N, v) \in \mathcal{QA}^N$, we consider two cases:

(i) Suppose that $(N,v) \in \mathcal{QA}^N$ is such that $v(\{i\}) \neq v(\{j\})$ for some $i,j \in N$. It must be that $\frac{v(\{h\})}{K} \neq \frac{1}{n}$ for some $h \in N$. Then, from (6.21) we obtain

$$g(\sum_{j\in N} v(\{j\}), v(N)) = g(\sum_{j\in N} v(\{j\}), v^N(N)),$$

which means that $g : \mathbb{R}^2 \to \mathbb{R}$ is a constant function with respect to its second argument for each $\sum_{j\in N} v(\{j\})$ since $v^N(N) = \sum_{j\in N} v(\{j\})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = g(x,y)$ for all $x,y \in \mathbb{R}$. Then (6.3) can be written as

$$\psi_i(N,v) = \frac{v(N)}{n} + \Big(v(\{i\}) - \frac{\sum_{j\in N} v(\{j\})}{n}\Big) f(\sum_{j\in N} v(\{j\})). \tag{6.22}$$

Next, pick any $a \in \mathbb{R}$, consider $(N,av) \in \mathcal{QA}^N$. By (6.22) applied to this game, we have

$$\psi_i(N,av) = \frac{av(N)}{n} + \Big(av(\{i\}) - \frac{\sum_{j\in N} av(\{j\})}{n}\Big) f(\sum_{j\in N} av(\{j\})).$$

By homogeneity, $\psi_i(N,av) = a\psi_i(N,v)$, which implies

$$f(\sum_{j\in N} v(\{j\})) = f(\sum_{j\in N} av(\{j\})).$$

Since $a \in \mathbb{R}$ can take any real number, this implies that f is a constant function on $\mathbb{R}$. Thus, let $f(\sum_{j\in N} v(\{j\})) = \beta$, where β is an arbitrary constant. Therefore, using (6.22) we have

$$\psi_i(N,v) = \frac{v(N)}{n} + \beta\Big(v(\{i\}) - \frac{\sum_{j\in N} v(\{j\})}{n}\Big), \tag{6.23}$$

which equals to $\beta ESD + (1-\beta)ED$.

(ii) Suppose that $(N,v) \in \mathcal{QA}^N$ is such that $v(\{i\}) = v(\{j\})$ for all $i,j \in N$. Then, by (6.3) we have $\psi_i(N,v) = \frac{v(N)}{n}$, which also coincides with (6.23).

Step 2. Consider any game $(N,v) \in \mathcal{G}^N$. Applying (6.23) to (N,v^N), we have

$$\psi_i(N,v^N) = \beta v(\{i\}) + \frac{1-\beta}{n} \sum_{j\in N} v(\{j\}).$$

Substituting this equation into (6.2) from Lemma 6.1, we obtain

$$\psi_i(N,v) = \psi_i(N,v^N) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]$$

$$= \beta v(\{i\}) + \frac{1-\beta}{n}\sum_{j\in N} v(\{j\}) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]$$
$$= \frac{v(N)}{n} + \beta v(\{i\}) - \frac{\beta}{n}\sum_{j\in N} v(\{j\})$$
$$= \beta ESD_i(N,v) + (1-\beta)ED_i(N,v).$$

The proof is completed. □

Proof of Lemma 6.3. Let ψ be a value on $\mathcal{G}^N$ satisfying efficiency, weak no advantageous reallocation, equal loss under separatorization, and weak desirability. Similar to the proof of Lemma 5.3, we can obtain that if a value ψ on $\mathcal{A}^N$ satisfies efficiency, weak no advantageous reallocation, and weak desirability, then it also satisfies anonymity.

From Lemma 6.1, efficiency and equal loss under separatorization imply (6.2), and thus $\psi_i(N,v) = \psi_i(N,v^N) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})] = \psi_{\pi(i)}(N,\pi v^N) + \frac{1}{n}[\pi v(N) - \sum_{j\in N} \pi v(\{\pi(j)\})] = \psi_{\pi(i)}(N,\pi v)$ since $\pi v^N = (\pi v)^N$ for any permutation π. Thus, ψ satisfies anonymity on $\mathcal{G}^N$. □

Proof of Theorem 6.6. It is clear that $\psi = \beta ESD + (1-\beta)ED$, $\beta \in [0,1]$, satisfies the six axioms. Conversely, let ψ be a value on $\mathcal{G}^N$ that satisfies the six axioms. As stated in Lemma 6.3 and Theorem 6.5, there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1-\beta)ED$. Moreover, weak desirability brings $\beta \geq 0$, and superadditive monotonicity brings $\beta \leq 1$. □

6.7 Conclusion

In this chapter, we have proposed the axiom of equal loss under separatorization, and have formalized the class of values satisfying equal loss under separatorization and efficiency. After that, we added other well-known axioms to characterize (i) the class of affine combinations of the ED and ESD values, (ii) the class of convex combinations of the ED and ESD values, (iii) the ED value, and (iv) the ESD value.

Similar to the discussion in Subsection 5.4.2, we can also employ a new axiom, called *balanced contributions under separatorization*, requiring that, for any two players, the effects of one of them becoming a separator on the payoff of the other, should be affected equally. So, whereas equal loss under separatorization considers the effect on the payoffs of two players of a third player becoming a separator, the axiom here compares the mutual effect on the payoffs of the two players becoming separators (similar as in the famous balanced contributions axiom). Without going into details, we mention that this axiom can be used to replace equal loss under separatorization in the results in Chapter 6.

To extend the results in Kongo (2018), Kongo (2019) characterizes the weighted (surplus) division values by weakening the nullified equal loss property into sign equal effect of players' nullification. This weakening axiom requires that a player's nullification affects all others' payoffs in the same direction (positive, 0, or negative). As shown in this chapter, equal loss under separatorization instead of the nullified equal loss property keeps the validity of the results in Kongo (2018). So, it might be worthwhile to explore whether it is possible to characterize the weighted (surplus) division values by a weaker variation of equal loss under separatorization.

Summary

This thesis consists of six chapters on cooperative game theoretic issues. Except Chapter 1, which is an introductory chapter, each of the other five chapters contains original results. The common denominator between almost all chapters is the emphasis on a specific proportionality principle in the allocation process of the worth that results from the cooperation of all players. Although the proportionality principle is quite common in the theory of resource allocation (especially in bankruptcy problems), it has been relatively ignored or neglected in cooperative game theory. This thesis convincingly fills this gap: by proposing new axioms to characterize either existing – but often neglected in the literature – proportional value, or new proportional values. The thesis explores a still underdeveloped area in cooperative game theory.

Chapter 2 studies the proportional division value for TU-games. The *proportional division (PD) value* allocates the worth of the grand coalition in proportion to the stand-alone worths of the players. We characterize this value in terms of some intuitive fairness criteria that are widely used in the value theory for TU-games, including *equal treatment of equals*, *monotonicity*, and *consistency*. Remarkably, proportional-balanced treatment, one of our axioms, reflects not only *equal treatment of equals* but also *unequal treatment of unequals*. Our monotonicity axioms are a relaxation of three existing axioms by adding restrictions on the stand-alone worths of the players. The consistency principle we adopted is the well-known projection consistency, which is used in axiomatizing the equal division (ED) value and the equal surplus division (ESD) value.

Chapter 3 identifies the value by extending the balanced cost reduction property from queueing problems, which are so-called *2-games*, to TU-games. The *balanced cost reduction property* states that the payoff of any player equals the sum of all changes in the payoffs of all other players if that player leaves the queueing problem. After extending the characterization result for queueing problems to the class of 2-games, we show that an extension of this axiom for general TU-games is incompatible with efficiency. However, a variation of this axiom is compatible with efficiency. This variation, called *weak balanced externalities*, requires that every player's payoff is the same *fraction* of its total externality inflicted on the other players. More specifically, this axiom and efficiency together characterize the *PANSC value*, which allocates the worth of the grand coalition proportional to the separable costs of the players. The PANSC value and the PD value are dual to each other. Following a similar idea as in Chapter 2, we further provide characterizations of the PANSC value

for the classes of general TU-games and two-player games.

In Chapter 4, assuming that the worth of the grand coalition is randomly divided into two parts, we define a value by allocating the two parts based on proportional and equal division methods respectively. Each part is a linear function with respect to the worths of all coalitions, so various functions give rise to various values. We axiomatize this family of values by employing efficiency, the balanced individual excess ratio property, continuity, weak additivity, anonymity, and no advantageous reallocation across individuals. Meanwhile, efficiency, the balanced individual excess ratio property, and linearity together characterize the family of affine combinations of the ED value and the ESD value. It is worth noting that a novel analytical approach is provided to deal with a generalization of this family by only imposing projection consistency. As it turns out, this exactly picks out the PD value and the affine combinations of the ED value and the ESD value. We also implement specific values in this family by a procedure based on a one-by-one formation of the grand coalition when considering all possible permutations.

Chapter 5 deals with a subfamily of values studied in Chapter 4. This family is introduced by taking into account that the surplus of the grand coalition is allocated proportionally to the stand-alone worths of the players. They are formalized by the affine combinations of the PD value and the new EPSD value. The *EPSD value* first assigns to every player the average of all stand-alone worths, and then allocates the remaining surplus among all players proportional to their stand-alone worths. We provide characterizations for this family of values as well as any member belonging to this family using two parallel axioms on a fixed player set based on player separatorization.

Chapter 6 investigates the convex combinations of the ED value and the ESD value based on separatorization introduced in Chapter 5. This family is a class of egalitarian values, but also a subfamily of values studied in Chapter 4. We extend the existing results in Ferrières (2017) and Kongo (2018) by employing equal loss under separatorization instead of the nullified equal loss property. Equal loss under separatorization requires that any player becoming a separator yields the same change in payoff for all other players. We derive the expression of values that satisfy this axiom and efficiency. Then, we show that these two axioms together with fairness and homogeneity characterize the class of affine combinations of the ED and ESD values. On the other hand, following Ferrières (2017) and Kongo (2018), we added other well-known axioms and characterized the ED value, the ESD value, and the classes of their affine and convex combinations.

This thesis contains new axiomatic characterizations of either allocation rules already studied in the literature or new allocation rules. It may open a way to new axiomatic studies and new solutions for applied problems. Although the thesis merely discusses possible applications of proportional values in cooperative games, it seems clear that economic or operations research applications would deserve a detailed study.

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